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A SEQUENTIAL-TRADE MICROSTRUCTURE MODEL WITH HETEROGENEOUS INFORMATION SETS

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Abstract

I present two cases of sequential trade models of market microstructure with heterogeneous *information partitions. First, the trader with information can infer the true value only under certain realizations. The main result is that the equilibrium outcomes look quite different from the literature, suggesting that the information assumptions may be a delicate issue to consider in these models. In the second example I adapt a non-partitional information approach from Geanakoplos (1989). When the informed investor does not observe this signal, she does not infer whether the value of the asset is low or high. Here positive spreads may not lead to trade.*

I. Introduction

In the market microstructure literature[1] several papers have emphasized the role of information asymmetries in determining the bid-ask spreads observed in stock markets. In particular Glosten and Milgrom's (1987) piece of work constitute the canonical model (usually referred as *sequential trade model*) to address this issue. Easley-O'Hara (1987) have used a multi-period version to study the dynamics of the spread and other problems related to adverse selection effects. In this case it is assumed that the p-informed trader or *p-insider* has an informational advantage over the market maker. Although this is a plausible assumption for several stock exchanges, it may not be necessarily the case in all

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of them. Given that the institutions acting as dealers are usually big financial institutions, they may also get information which is not shared ex-ante by the rest of the market participants. The Glosten-Milgrom paper never assumed this in fact. Hence it is an open question the characterization of the equilibrium bid-ask spread given a different assumption on information sets.

In order to answer this question I present two examples of a two-period version of the sequential-trade models of the type described above. Here the p-insider does not have full information about the realization of the payoffs of the asset. Instead the market maker may possess a different information sets relative to the p-insider.[2] In the first example the p-insider does not have informational advantage all the time. I first assume that the p-insider only knows the true value when this is the lowest possible. For any other realization the p-insider is unable to make any distinction among all those values. On the other hand the market maker posts bid and ask quotes that are *measurable* with respect to the information partition that she has. This implies that the bid and the ask must be the same as long as the dealer cannot differentiate between two or more realizations of the value of the asset. Therefore the p-insider can infer the true value of the security as long as the intersection of the elements of her partition and the elements of the dealer's partition gives a singleton. This is because the p-insider could observe her own initial partition as well as the partition induced by the bid and ask quotes. Then in equilibrium the asks and bids are such that the p-insider is able to differentiate two out of four possible realizations, but not the other two.

This clearly depends crucially on the partitions of each agent. I show this by changing the information structure of both traders. I then assume, for the same partition of the p-insider, that the dealer knows perfectly the value of the asset if the realization is the highest possible. In this case there will be no trade if the value of the asset is the highest since both types of agents know the true value for that realization. The reason is that the p-insider infers perfectly the true value when this is the highest one from the bid-ask distribution. In this second case the p-insider buys the security in some state where before, in the first case, she was not buying it. If the information sets are interchanged then the result is that in the lowest possible value there will be no trade while the p-insider will sell under states where in the first case of the first example the p-insider was not trading. All this demonstrates that the equilibrium that arises in these types of models is very dependent of the assumption about the availability of information for each agent. In a sense this would say that the sequential trade models are not very robust to the information assumptions. However this also confirms that *speculation* can also arise in situations where the p-insider does not have *perfect* signals about the true realization of the value of the security. As the cases 2 and 3 of the first example show, the p-insider may buy or sell in states where she does not fully identify the true value. This is true provided that the p-insider still observes an information set which is smaller than the one observed by the dealer. I provide a generalization of this example.

I then present a second example. In this case I follow closely Geanakoplos (1989). The p-insider possesses information sets that are *not* partitional. To see the idea, assume a standard sequential trade model of the Easley-O'Hara type, where there are only two possible realizations. I assume that the p-insider infers correctly the true realization whenever this is the low value. However if the realization is the high value the signal does not convey any information. In other words, if the high value is realized, then the p-insider thinks that either the low or the high are still possible. This may be interpreted in different ways. Nevertheless it is natural to have this as an approximation of some type of pessimistic behavior.[3] This could have other interpretations. In any case the main result is that, with this assumption, there is a bid-ask spread even though there is no *adverse selection* effect in one of the states. This is because when the true realization is high the information set faced by the p-insider is exactly the same as the one observed by the market maker. Therefore the intersection of those two gives the same set. On the other hand if the true realization is the low value therefore there is adverse selection. The quotes must still be measurable with respect to the market maker's information. Since in this second example the dealer has no information about the true value, then the quotes must be the same regardless of the realization. Since in one of the states there is adverse selection, then a bid-ask spread is generated in the usual way. The combination of this and the measurability of the quotes gives the result. The important point is that non-partitional information implies that it may suffice to have adverse selection under only one possible state of the world to generate a bid-ask spread. Another implication of these assumptions is that the speculative trading will be lower than in the standard models. In the second example the p-insider sells only when the asset has a low value, but she *never* buys the security. This is again consistent with a intuition of pessimistic behavior.

Some related work is presented by Foster and Viswanathan (1996) and by Madrigal and Scheinkman (1997). In the first case there is a set of heterogeneously informed p-insiders. The market maker in that case has still to infer the information through trading. This is an example of an *information aggregation* problem, where the specialist is the one "in charge" of aggregating different types of information. In the second paper, the main result is the presence of a discontinuity of the equilibrium price function which is interpreted as *price crashes*, although there is no dynamics involved. This is also the outcome of an information aggregation process by the market maker. However none of these two papers consider the case where the specialist and the p-insider have different (though not necessarily better) information.

Section II describes the first example. Section III gives the generalization of the example in Section II. Section IV presents the second example with non-partitional information sets. Section V gives an interpretation for each case as well as their implications. Finally, Section VI gives some concluding remarks and directions for future research.

II. A First Example

2.1 The environment

There are two periods, $t = 0$ and $t = 1$. There is a unique asset whose true value is only revealed completely at the beginning of period 1. The asset trading session takes place previous to that moment. The true value V is considered random at the beginning of period 0 and it could take four values. Without loss of generality I assume that $V \in \Omega I = \{1, 2, 3, 4\}$. Denote as ω the typical element of the ΩI. The common priors are given by Pr $[V = i] = q_i$, for $i = 1, 2, 3, 4$ with $q_i \in (0, 1]$ and $\Sigma_i q_i = 1$.

There are three types of market participants. There is a measure ρ of liquidity or *noise* traders. These agents have no information about the environment. Therefore they buy one unit of the asset with probability ε and sell one unit with probability $(1 - \varepsilon)$, with $0 < \varepsilon < 1$. The reader is referred to the classic paper by Milgrom and Stokey (1982) for a justification of this assumption. There is a measure $(1 - \rho)$ of risk neutral *ρ-informed* traders or *p-p-insiders*. These agents have some knowledge about the value of the asset through some partition of the state space, although this is not necessarily the finest one (This is why I call them p-informed). I assume a measure one of risk neutral *market makers*. They also have some information partition once the true value is chosen by Nature.

This type of asymmetry in the partitions faced by p-insiders and market makers could seem unnatural. There are however situations that can be interpreted as the partitions in this paper. For example, it is often the case that dealers possess better expertise to interpret more accurately information about the macroeconomy than investors, although this may not be true in the case of big institutional traders. Situations like the mentioned here seem to fit well with the idea that dealers potentially have better information than investors. This implies that, for example, dealers interpret perfectly certain macroeconomic shocks, while p-informed investors do not. The states in which dealers are better informed that dealers in the model can be then interpreted as fundamental environment shocks that only dealers interpret perfectly. In the next subsection I present the first case where the p-insider only knows the true value when this is the lowest possible. I will consider other possible information sets in subsequent subsections.

2.2 Case 1

The sequence of events is as follows. At the beginning of period 0 Nature chooses a determined value for ω. This value is not known to neither the dealer nor the p-insider. However each of them have information partitions of $\{1, 2, 3, 4\}$ that reveal some information to each of the informed participants, although not necessarily in a perfect way. In this first case I assume the following partitions for the dealer and for the p-insider.

$$\mathcal{B} = \{\{1, 2\}, \{3, 4\}\}$$

for the p-dealer and

$$\mathcal{F} = \{\{1\}, \{2, 3, 4\}\}$$

for the p-insider. These information partitions are common knowledge for all agents. This is interpreted as follows. The p-informed trader only has perfect information whenever the true value of the asset is the lowest possible, called for simplicity the ***bankruptcy state*** for the firm. This situation can then be interpreted as the case where the p-insider knows only whether the firm could go bankrupt or not. However this same p-insider does not know otherwise how well the firm will do. On the other hand the dealer does not know if the firm goes bankrupt or not. However the dealer has some information whether the economy performs *well* or *not well*. Here I identified the state in which the economy does well as the element $\{3, 4\}$. The set $\{1, 2\}$ is identified as the state in which the macroeconomy suffers an adverse shock. In this last situation the bankruptcy state is included but not differentiated with the state 2 element. The negative shock is on the other hand well understood by the dealer but not by the p-insider. This only can know whether the firm goes bankrupt or not (state 1). However the p-informed investor does not know anything else about the reality of the economy. Hence the dealer confuses states 1 and 2, while the p-informed agent confuses 2 with 3 and 4.

After knowing this the dealer posts bids and asks and trade takes place under these quotes as in the usual sequential trade models. Given risk-neutrality, the ask is the expected value of the asset conditional to a buy observed by the dealer and the element of $\mathcal{B}$ revealed to the dealer. The bid is the expected value of the asset conditional to a sell observed by the market maker and the element of $\mathcal{B}$ revealed to the dealer. In the next subsection I describe the equilibrium conditions and present the results for this paper.

2.3 Equilibrium behavior in case 1

Denote A the ask and B the bud posted by the dealer. Denote ω a typical element of $\{1, 2, 3, 4\}$. Following Glosten and Milgrom the p-insider buys if $Z(\omega) > A(\omega)$ where

$$Z(\omega) = E[V \mid H, A, B](\omega)$$

and where $H \in \mathcal{F}$. Similarly the p-insider sells if $Z(\omega) < B(\omega)$, where Z is defined as above. I make the following assumption about the p-insiders.

Assumption 1. The p-insiders will *not* buy if $Z(\omega) = A(\omega)$. Similarly the p-insiders will *not* sell if $Z(\omega) = B(\omega)$.

On the other hand the bids and asks are given by

$$A(\omega) = E[V \mid J, buy](\omega)$$
$$B(\omega) = E[V \mid J, sell](\omega)$$

where $J \in \mathcal{B}$.

The computation of equilibria will require to consider the actual realization of ω. Suppose first that Nature chooses $\omega = 1$, that is, the firm going bankrupt. Therefore the p-insider knows this perfectly. This implies that $Z(1) = 1$. The set containing 1 for the dealer is the set $\{1, 2\}$. Therefore the ask and the bid are given by

$$\begin{aligned} A(1) &= E[V \mid \{1, 2\}, buy](1) \\ &= P[V = 1 \mid \{1, 2\}, buy]\,1 + P[V = 2 \mid \{1, 2\}, buy]\,2 \end{aligned}$$

$$\begin{aligned} B(1) &= E[V \mid \{1, 2\}, sell](1) \\ &= P[V=1 \mid \{1, 2\}, sell]\,1 + P[V=2 \mid \{1, 2\}, sell]\,2 \end{aligned}$$

This is because $P[V = i \mid \{1, 2\}, buy] = P[V = i \mid \{1, 2\}, sell] = 0$, for $i = 3, 4$. The expressions for the bid and the ask imply that $A(1) \in (1, 2)$ and $B(1) \in (1, 2)$, since $q_i > 0$ for all i. To get the values for $A(1)$, $B(1)$ I must compute the conditional probabilities using Bayes rule. First:

$$\begin{aligned} &P[V = 1 \mid \{1, 2\}, buy] \\ &= \frac{P[\{1, 2\}, buy \mid V=1]\,P[V=1]}{P[\{1, 2\}, buy \mid V=1]\,P[V=1] + P[\{1, 2\}, buy \mid V=2]\,P[V=2]} \end{aligned}$$

Note that if $V=1$ or $V=2$, then V is in $\{1, 2\}$. Therefore:

$$\begin{aligned} &P[\{1, 2\}, buy \mid V=1] \\ &= P[buy \mid V=1] \end{aligned}$$

and

$$\begin{aligned} &P[\{1, 2\}, buy \mid V=2] \\ &= P[buy \mid V=2] \end{aligned}$$

I now compute the ask. If $V=1$ is true, clearly the dealer would infer that $Z=1$. This is due to the common knowledge assumption. However, since $A \in (1, 2)$ then $A > V$. This implies that if $V=1$ there is no p-insider buying the asset. As a consequence $P[buy \mid V=1] = \rho\varepsilon$. On the other hand, if $V=2$ were true then the dealer infers that Z is not 1 but 2. This is because the dealer knows that if $V=2$,

the p-insider would observe $\{2, 3, 4\}$. However, since $A(2) \neq A(3) = A(4)$ (shown below) then the p-insider would correctly infer that $V=2$. As long as $Z^{(2)} > A(1)$ the p-insider would buy the asset and if the contrary is true the p-insider will not buy the asset (given that $V=2$). However, note that

$$Z^2 = 2$$
$$A(1) \in (1, 2)$$

This implies that $Z^2 > A(1)$. Then the p-insider will buy if $V=2$. As a consequence: $P[buy \mid V=2] = \rho\varepsilon + (1-\rho)$. This implies that:

$$\begin{aligned} &P[V=1 \mid \{1,2\}, buy] \\ &= \frac{\rho\varepsilon q_1}{\rho\varepsilon q_1 + [\rho\varepsilon + (1-\rho)]\,q_2} \end{aligned}$$

In a similar way:

$$\begin{aligned} &P[V=2 \mid \{1, 2\}, buy] \\ &= \frac{[\rho\varepsilon + (1-\rho)]\,q_2}{\rho\varepsilon q_1 + [\rho\varepsilon + (1-\rho)]\,q_2} \end{aligned}$$

and then

$$A^*(1) = \frac{\rho\varepsilon q_1 + 2\,[\rho\varepsilon + (1-\rho)]\,q_2}{\rho\varepsilon q_1 + [\rho\varepsilon + (1-\rho)]\,q_2} \tag{1}$$

This is the first equilibrium ask price for the realization $\omega = 1$.

To get the bid price one must compute the conditional probabilities $P[sell \mid V=1]$ and $P[sell \mid V=2]$. If $V=1$ the dealer knows that $Z=1$. This implies that $Z < B(1)$[4]. Therefore the p-insider *will* sell the security if $V=1$. Hence $P[sell \mid V=1] = \rho(1-\varepsilon) + (1-\rho)$. On the other hand, if $V=2$, the dealer knows that $Z = Z^{(2)}$ given above. Since $Z^2 > 2$ and $B(1) < 2$, then $B(1) < Z^{(2)}$. This implies that if $V=2$ the dealer knows that only the liquidity traders would sell the security. This implies that $P[sell \mid V=2] = \rho(1-\varepsilon)$. Therefore:

$$B^*(1) = \frac{[\rho(1-\varepsilon) + (1-\rho)]\,q_1 + 2\rho(1-\varepsilon)\,q_2}{[\rho(1-\varepsilon) + (1-\rho)]\,q_1 + \rho(1-\varepsilon)\,q_2} \tag{2}$$

In order to assure that these are the actual equilibrium quotes for $\omega = 1$, I must show that $A^*(1) = A^*(2)$, and that $B^*(1) = B^*(2)$. This is because the quotes must be $\mathcal{B}$ – measurable functions of $\{1, 2, 3, 4\}$. This measurability condition is implied by the fact that the dealer should not be able to differentiate between $\omega = 1$ and $\omega = 2$. Otherwise she could infer the true state of the world.

I must now compute the equilibrium quotes $A^*(2)$ and $B^*(2)$ and compare these with the obtained prices. Suppose now that $\omega = 2$. By the same argument as above, it is true that

$$\begin{aligned} A(2) &= E[V \mid \{1, 2\}, buy](2) \\ &= P[V=1 \mid \{1, 2\}, buy]1 + P[V=2 \mid \{1, 2\}, buy]2 \end{aligned}$$

But this is the same formula as $A(1)$. Therefore $A^*(2) = A^*(1)$. In a similar way

$$\begin{aligned} B(2) &= E[V \mid \{1, 2\}, sell](2) \\ &= P[V=1 \mid \{1, 2\}, sell]1 + P[V=2 \mid \{1, 2\}, sell]2 \end{aligned}$$

Therefore $B^*(2) = B^*(1)$. Then this model satisfies the measurability condition for $\omega = 1$ and $\omega = 2$. Note that when the true value is 2 the p-insider is not perfectly informed anymore. However, due to the heterogeneous partitions, the p-insider still buys the asset under this event. This implies that the speculative motive for transactions is still present even though the information possessed by the p-insider may not be better than the information that the market maker has.

Assume now that $\omega = 3$ or $\omega = 4$. In this case the quotes are given by the following:

$$\begin{aligned} A(3) &= E[V \mid \{3, 4\}, buy](3) \\ &= P[V=3 \mid \{3, 4\}, buy]\,3 + P[V=4 \mid \{3, 4\}, buy]\,4 \\ &= E[V \mid \{3, 4\}, buy](4) \\ &= A(4) \end{aligned}$$

Again $A(3) = A(4)$ and then measurability is automatically satisfied. Compute now

$$P[V=3 \mid \{3, 4\}, buy] = \frac{P[\{3, 4\}, buy \mid V=3]\ P[V=3]}{P[\{3, 4\}, buy \mid V=3]\ P[V=3] + P[\{3, 4\}, buy \mid V=4]\ P[V=4]}$$

By the same reason as before I must just care of the probabilities $P[buy \mid V=3]$ and $P[buy \mid V=4]$. Now, if $\omega = 3$, or $\omega = 4$ were true then the dealer knows that Z is given by

$$Z^{(3)} = Z^{(4)} = \frac{3q_3 + 4q_4}{q_3 + q_4}$$

This *must* be true since the function Z is $\mathcal{F}$ - measurable. On the other hand I do not include the value $V = 2$. This is because the p-insider can differentiate between 2 and $\{3, 4\}$ since the quotes will be different. The only problem is that the p-insider cannot differentiate between 3 and 4. This is the justification of this

formula. Note then that $Z^{(3)}$ and $Z^{(4)}$ are both in (2, 4). On the other hand, with all our assumptions $A(3) \in (3, 4)$. Now I do not have a straightforward relationship between $A(3)$ and $Z^{(3)}$ and between $A(4)$ and $Z^{(4)}$. This would complicate the computation of the conditional probabilities. However I will show that I could not have $A(j) < Z^{(j)}$ for $j = 3, 4$ later on. In fact I show that $A(j) = Z^{(j)}$. This will imply inaction from the part of the p-insider.

I must now get the values of $P[buy \mid V=3]$. Note that there will be ρ liquidity traders buying with probability ε. Assume first that $A(3) < Z^{(3)}$. If this is the case then the p-insider will buy. In this case

$$P[buy \mid V=3] = \rho\varepsilon + (1 - \rho)$$

Since $A(3) = A(4)$ and $Z^{(3)} = Z^{(4)}$ then

$$P[buy \mid V=4] = \rho\varepsilon + (1 - \rho)$$

Therefore $P[buy \mid V=3] = P[buy \mid V=4]$. This implies that

$$P[V=3 \mid \{3, 4\}, buy] = \frac{q_3}{q_3 + q_4}$$

$$P[V=4 \mid \{3, 4\}, buy] = \frac{q_4}{q_3 + q_4}$$

Hence both $A(3)$ and $A(4)$ are equal to

$$\frac{3q_3 + 4q_4}{q_3 + q_4}$$

However this is exactly $Z^{(3)} = Z^{(4)}$. Therefore the initial conjecture that $A(3) < Z^{(3)}$ is not true under our assumptions. This is also true for $A(4) < Z^{(4)}$ since it is the same argument. I have shown the following result.

Lemma 1. *It is impossible to have* $A(\omega) < Z^{(\omega)}$ *for each* $\omega = 3, 4$. *In fact* $Z(\omega) = A(\omega)$ *for each* $\omega = 3, 4$.

Therefore we should have that $A(\omega) = Z^{(\omega)}$, $\omega = 3, 4$. Then $P[buy \mid V=3]$ and $P[buy\ V=4]$ are both equal to $\rho\varepsilon$. Note that this still implies that

$$A^*(3) = \frac{3q_3 + 4q_4}{q_3 + q_4} \tag{3}$$

$$A^*(4) = \frac{3q_3 + 4q_4}{q_3 + q_4}$$

These are the equilibrium ask quotes whenever $\omega = 3$ and $\omega = 4$. Note that under these two states the p-insiders will *not* buy the security, although in a more standard sequential trade model the p-insider would buy at least for the realization $\omega = 4$.

The bid quotes can be calculated in a similar fashion. The bids for $\omega = 3$ and $\omega = 4$ are given by the following expressions:

$$\begin{aligned} B(3) &= B(4) \\ &= P\,[V{=}3 \mid \{3, 4\}, \mathit{sell}]\,3 + P\,[V{=}4\ \{3, 4\}, \mathit{sell}]\,4 \end{aligned}$$

This again implies to calculate $P\,[\mathit{sell} \mid V{=}3]$ and $P\,[\mathit{sell} \mid V{=}4]$. In each of the cases, with probability $\rho\,(1-\varepsilon)$ a liquidity trader will sell the security. On the other hand, if $V{=}3$ were true then the value $Z^{(3)} = Z^{(4)}$ is given by the former formula, that is:

$$\frac{3q_3 + 4q_4}{q_3 + q_4}$$

To get the probabilities, suppose that $Z^{(\omega)} < B(\omega)$ for $\omega = 3, 4$. This would imply that

$$\begin{aligned} &P\,[\mathit{sell} \mid V{=}3] \\ &= P\,[\mathit{sell} \mid V{=}4] \\ &= \rho\,(1-\varepsilon) + (1-\rho) \end{aligned}$$

But then

$$P\,[V{=}3 \mid \{3, 4\}, \mathit{sell}] = \frac{q_3}{q_3 + q_4}$$

$$P\,[V{=}4 \mid \{3, 4\}, \mathit{sell}] = \frac{q_4}{q_3 + q_4}$$

Therefore $B(3) = B(4)$ are both equal to

$$\frac{3q_3 + 4q_4}{q_3 + q_4}$$

But then $B(\omega) = Z^{(\omega)}$ for $\omega = 3, 4$. Then the initial guess $Z^{(\omega)} < B(\omega)$ was not correct. Then $Z^{(\omega)} = B(\omega)$ for $\omega = 3, 4$. As a consequence both probabilities $P\,[\mathit{sell} \mid V{=}3]$ and $P[\mathit{sell} \mid V{=}4]$ are equal to $\rho\,(1-\varepsilon)$. This implies:

$$P\,[V{=}3 \mid \{3, 4\}, \mathit{sell}] = \frac{q_3}{q_3 + q_4}$$

$$P\,[V{=}4 \mid \{3, 4\}, \mathit{sell}] = \frac{q_4}{q_3 + q_4}$$

Therefore $B^*(3)$ and $B^*(4)$ are both equal to:

$$\frac{3q_3 + 4q_4}{q_3 + q_4} \tag{4}$$

Therefore $B^*(\omega) = A^*(\omega)$, $\omega = 3, 4$. There is no trade in these two states. Therefore I summarize this result in the following proposition:

Proposition 2. *There is an equilibrium in the first case presented in Section II. The equilibrium bid and ask quotes are given by the expressions 1, 3, 2, 4. Moreover the p-insider will sell the security only when $\omega = 1$ and buy when $\omega = 2$. There is no trade in either state $\omega = 3$ or $\omega = 4$.*

2.4 How *robust* are the examples to the information specification? Cases 2 and 3

In the last section the main result says that, for the information specification given in the first sub-section, the p-insider would not trade for speculative reasons even if the true value is high enough. I show next that this is not *robust* in the sense that, for a different information specification, very different results arise. Suppose the following partitions

$$\mathcal{B} = \{\{1, 2, 3\}, \{4\}\}$$

for the dealer and

$$\mathcal{F} = \{\{1\}, \{\{2, 3, 4\}\}$$

for the p-insider. This is the case where the dealer has better a-priori information than the p-insider whenever the realization is the highest value. It can be shown that the following is an equilibrium. Its proof is in the appendix.

Proposition 3. *There is an equilibrium in the economy presented in Section II under the last information specification. The equilibrium bid and ask quotes are given by the following expressions:*

$$A^*(\omega) = \frac{q_1\rho\varepsilon + [\rho\varepsilon + (1-\rho)]\,[2q_2 + 3q_3]}{q_1\rho\varepsilon + [\rho\varepsilon + (1-\rho)]\,[q_2 + q_3]}$$

$$\omega = 1, 2, 3$$

$$B^*(\omega) = \frac{q_1\,[\rho\,(1-\varepsilon) + 1\text{-}\rho] + \rho\,(1-\varepsilon)\,[2q_2 + 3q_3]}{q_1\,[\rho\,(1-\varepsilon) + 1\text{-}\rho] + \rho\,(1-\varepsilon)\,[q_2 + q_3]}$$

$\omega = 1, 2, 3$

$$A^*(4) = B^*(4) = 4$$

Moreover the p-insider will sell the security only when $\omega = 1$. The p-insider buys when $\omega = 2$ and $\omega = 3$. There is no trade in state $\omega = 4$.

This re-emphasizes the dependence of the equilibrium outcome on the assumptions about the information availability in these sequential trade models. In fact I can get exactly the opposite result if I change the information sets as follows. Suppose now the following partitions.

$$\mathcal{B} = \{\{1\}, \{2, 3, 4\}\}$$

for the dealer and

$$\mathcal{F} = \{\{1, 2, 3\}, \{4\}\}$$

for the p-insider. I state the following proposition whose proof is in the appendix.

Proposition 4. *There is an equilibrium in the economy presented in Section II under the last information specification. The equilibrium bid and ask quotes are given by the following expressions:*

$$A^*(\omega) = \frac{(2q_2 + 3q_3)\,\rho\varepsilon + 4q_4\,[\rho\varepsilon + 1-\rho]}{(q_2 + q_3)\,\rho\varepsilon + q_4\,[\rho\varepsilon + 1-\rho]}$$

$\omega = 2, 3, 4$

$$B^*(\omega) = \frac{(2q_2 + 3q_3)\,[\rho\,(1-\varepsilon) + 1-\rho] + 4q_4\rho\,(1-\varepsilon)}{(q_2 + q_3)\,[\rho\,(1-\varepsilon) + 1-\rho] + q_4\rho\,(1-\varepsilon)}$$

$\omega = 2, 3, 4$

$$A^*(1) = B^*(1) = 1$$

Moreover the p-insider sells the security when $\omega = 2$ or $\omega = 3$ The p-insider buys only when $\omega = 4$. There is no trade in state $\omega = 1$.

This clearly shows that just switching the information sets give totally opposite results. I refer to Section IV for further comments and interpretations.

III. The General Model With Information Partitions

Some of the results stated above for the specific examples could be generalized under certain conditions. Suppose now that $\omega \in \Omega I = \{\omega_1, \omega_2, ..., \omega_p\}$. Assume without loss of generality that $\omega_1 > 0$ and $\omega_{i+1} > \omega_i$. Let $q_i \equiv \Pr[V = \omega_i]$. Let also $\mathcal{F}$ be the partition of the market maker. For notational purposes, assume that $\mathcal{F} = \{D_1, ..., D_n\}$, where $n < p$. Here D_i is a subset of ΩI, such that $D_i \cap D_j = \emptyset$ for $i \neq j$ and $\cup_i D_i = \Omega I$. On the other hand define $\mathcal{B} = \{I_1, ..., I_m\}$, with $m \leq p$ the partition for the p-insider. Take any ω in ΩI. Define $M(\omega) \equiv D_i(\omega) \cap I_j(\omega)$, some i and j. This is the set that is the intersection of the elements of the partitions $\mathcal{F}$ and $\mathcal{B}$ containing the state s. Therefore I can state the following result. The proof is also in the appendix.

Proposition 5. *Suppose that $M(\omega) = \{\omega\}$. Then the p-insider knows this state with probability 1 whenever it is realized. Suppose instead that $M(\omega)$ is not a singleton. Suppose furthermore that $D_i(\omega) \subset I_j(\omega)$ strictly, where both sets are such that $M(\omega) = D_i(\omega) \cap I_j(\omega)$. Then, for every element x in $M(\omega)$ the bid-ask spread is 0 and there is no speculative trade whenever $\omega = x$.*

To prove this, I use the following lemma also shown in the appendix.

Lemma 6. *The p-insider forms her own expected value $Z^{(\omega)}$ using a partition induced by her own initial partition and the dealer's partition.*

The last proposition just constitutes a generalization of proposition 2. Note that in the first case of the first example the set $\{3, 4\}$ is included in $\{2, 3, 4\}$, the first one being an element of the dealer's partition and the second one being an element of the p-insider's partition. On the other hand one may ask what would be the generalizations of the other two cases in the first example. Those are possible only under some special assumptions about the priors q_ω. The following proposition summarizes the main findings. The proof can be found in the appendix.

Proposition 7. *Let $M(s)$ defined as before. Suppose it is not a singleton. Moreover, if $D_i(\omega)$ and $I_j(\omega)$ are such that*

$$\frac{\left[\sum_{j:\omega_j \in M} q_j\omega_j\right]}{\left[\sum_{j:\omega_j \in M} q_j\right]} > \frac{\left[\sum_{\substack{j:\omega_i \in D_i \\ \omega_i \notin I_j}} q_i\omega_i\right]}{\left[\sum_{\substack{i:\omega_i \in D_i \\ \omega_i \notin I_j}} q_i\right]} \qquad (**)$$

If the p-insider does not buy under any state ω' such that $\omega' \in D_i \backslash M$, then the p-insider will buy the asset under any x in M. Moreover the p-insider does not sell the security in M.

The condition (**) says that the expected value conditional to the set M is greater than the expected value conditional to the set $D_i \backslash M$. This condition is satisfied in case 2 of the first example. This then says that it suffices to have a higher conditional expected value under these realizations to "induce" the p-insider to buy the asset (if the p-insider does not buy under the realizations in the set $D_i \setminus M$).

There is also a *generalization* for the case 3 of the first example. The interpretation of this proposition is also symmetric from the one of proposition 7 (the proof is left to the reader).

Proposition 8. *Let $M(\omega)$ defined as before. Suppose it is not a singleton. Moreover, if $D_i(\omega)$ and $I_j(\omega)$ are such that*

$$\frac{\left[\sum_{j:\omega_j \in M} q_j \omega_j\right]}{\left[\sum_{j:\omega_j \in M} q_j\right]} < \frac{\left[\sum_{\substack{i:\omega_i \in D_i \\ \omega_i \notin I_j}} q_i \omega_i\right]}{\left[\sum_{\substack{i:\omega_i \in D_i \\ \omega_i \notin I_j}} q_i\right]} \qquad (***)$$

If the p-insider does not sell under any state ω' such that $\omega' \in D_i$, $\omega' \notin M$ then the p-insider will sell the asset under any x in M. Moreover the p-insider does not buy the security in M.

IV. Non-Partitional Information Structures

4.1 The model

Consider now the following situation. There are two periods and the asset has a future value that could only take two values, $\{1, 2\}$. I still interpret $V=1$ as the case where the firm goes bankrupt. The timing is as in Section II. The notation is similar as in the last section. There is a measure ρ of liquidity traders who buy with probability ε and sell with probability $(1-\varepsilon)$. Assume that $\rho \in (0, 1)$ and $\varepsilon \in (0, 1)$. There is a measure $(1-\rho)$ of p-insiders. The information structure is as follows. The market maker does not have any way of differentiating between the two values. Therefore the information set that the dealer faces is given by:

$$\mathcal{B} = \{1, 2\}$$

The structure of the p-insider's information is somehow non-standard. I follow closely Geanakoplos (1989) here. If the true realization is $V=1$ then I assume that the p-insider can know this entirely. That is, if $V=1$, the set observed by the p-insider is $\{1\}$. However, if the true realization is $V=2$ the p-insider does not have any way of figuring this out. This implies that if $V=2$ the p-insider observes $\{1, 2\}$, the entire state space.

There are several interpretations for this assumption. One possible way to look at this is by having an investor who could only know the truth in the state in which the firm goes bankrupt. For example, she receives some credible news about the firm going bankrupt. However, if the p-insider does not receive any bad news she *cannot* infer that the firm *does not* go bankrupt. The ignorance of news does not preclude the bankruptcy possibility, at least from the p-insider's point of view. Notice the difference with the Section II model. In that case the signal received by the p-insider was clear enough to know if the firm does or does not go bankrupt, i.e., whether $V=1$ or $V>1$. In this second case the signal observed by the p-insider is completely noisy. If the true state is $V=1$ then the signal conveys true information to the p-insider. If the true state were $V=2$ then the signal may be less precise. Hence the p-insider still thinks that it is possible that the firm goes into bankruptcy.

Another interpretation is that the p-insider is *boundedly rational* in the following sense. The information processing of the signal by the p-insider is such that she could only infer correct information from one of the values of the signal, the bad state value. However the p-insider is not able to get any information whenever she receives a good signal[5]. This is what in the introduction I call *pessimistic* behavior. In any case this is the assumption used in the original work by Geanakoplos to model boundedly rational players in games. I study the implications of this different information structure for the formation of quotes.

4.2 Equilibrium analysis

I also suppose that assumption 1 given in the last section is true. The conditions characterizing an equilibrium are the same as in the last section and as in the standard sequential trade models. The ask in either state is $A=E\,[V \mid buy]$, which is equal to $P\,[V=1 \mid buy] + 2\,[V=2 \mid buy]$. Note again that in equilibrium the market maker does not differentiate between $V=1$ and $V=2$. That is why the quotes must be the same regardless of the realization of the true value. I compute the conditional probabilities in the usual fashion.

$$P[V=1 \mid buy] = \frac{P[buy \mid V=1]\, P[V=1]}{P[buy \mid V=1]\, P[V=1] + P[buy \mid V=2]\, P[V=2]}$$

$$P[V=2 \mid buy] = \frac{P[buy \mid V=2]\, P[V=2]}{P[buy \mid V=1]\, P[V=1] + P[buy \mid V=2]\, P[V=2]}$$

I consider again the two possible realizations $V=1$ and $V=2$ separately. Suppose that the true realization is $V=1$. If this is the case then the p-insider knows exactly this realization. Then the value of Z as defined in Section II is just $Z=1$. Note that the ask A is strictly greater than 1 and strictly below 2. Therefore if $\omega=1$ is true then the p-insider will ***not*** buy the asset. Then $P\,[buy \mid V=1] = \rho\varepsilon$. Suppose now that the true realization is V=2. In this case the p-insider does not know the true state. Therefore the value of Z under this assumption is $Z^{(2)}=q_1+2q_2$. I claim that there is an equilibrium where the p-insider does not buy, given that whenever $Z^{(2)}=A$ then the p-insider decides not to buy the security. Suppose that this is the case. Therefore the probability $P\,[buy \mid V=2\,]$ is equal to $\rho\varepsilon$, which is the same as the probability $P\,[buy \mid V=1]$. If this is the case, then the conditional probabilities are given by $P\,[V=\omega \mid buy] = q_\omega$, with $\omega = 1, 2$. Then this implies that the equilibrium ask price is A, equal to q_1+2q_2, which at the same time equalizes $Z^{(2)}$. On the other hand, bids are given by

$$\begin{aligned} B &= E\,[V \mid sell] \\ &= P[V=1 \mid sell] + 2P[V=2 \mid sell] \end{aligned}$$

Note again that $B \in (1, 2)$. This implies that if $\omega = 1$ then the p-insider will sell the security since $Z^{(1)}=1$. Therefore

$$P[sell \mid V=1] = \rho\,(1-\varepsilon) + (1-\rho)$$

If V=2, I again need to know whether the p-insider will sell or not. I guess that there is an equilibrium where the p-insider does not sell if V=2. Suppose that this is the true value. Then the p-insider has $Z^{(2)}=q_1+2q_2$ as the expected value of the asset. If the p-insider does not sell is because $Z^{(2)} > B$. Suppose this is true. Therefore

$$P[sell \mid V=2] = \rho\,(1-\varepsilon)$$

This implies the following expressions for the conditional probabilities are:

$$P[V=1 \mid sell] = \frac{q_1\,[\rho\,(1-\varepsilon) + (1-\rho)]}{q_1\,[\rho\,(1-\varepsilon) + (1-\rho)] + q_2\rho\,(1-\varepsilon)}$$

$$P[V=2 \mid buy] = \frac{q_2\rho\,(1-\varepsilon)}{q_1\,[\rho(1-\varepsilon) + (1-\rho)] + q_2\rho\,(1-\varepsilon)}$$

and the bid is

$$B = \frac{q_1\,[\rho\,(1-\varepsilon) + (1-\rho)] + 2q_2\rho\,(1-\varepsilon)}{q_1\,[\rho\,(1-\varepsilon) + (1-\rho)] + q_2\rho\,(1-\varepsilon)}$$

Therefore:

$$q_1 + 2q_2 > \frac{q_1\,[\rho\,(1-\varepsilon) + (1-\rho)] + 2q_2\rho\,(1-\varepsilon)}{q_1\,[\rho\,(1-\varepsilon) + (1-\rho)] + q_2\rho\,(1-\varepsilon)}$$

which is true if and only if:

$$q_1\,[\rho\,(1-\varepsilon) + (1-\rho)]\,[q_1+2q_2-1] + q_2\rho\,(1-\varepsilon)\,[q_1+2q_2-2] > 0 \quad ((*))$$

This is the condition to have an equilibrium where the p-insider only sells in the event of V=1. Note that this is true when $\rho = 0$. By continuity I can state that this equilibrium exists as long as the number of liquidity traders is small enough (although positive).

Note that this implies a positive bid-ask spread. However this model predicts that the p-insider would only sell the asset as long as she knows that the true value is low (or that the firm is going bankrupt). However she will *never* buy it (provided that inaction is true whenever the p-insider is indifferent). On the other hand the bid-ask spread exists *regardless of* the realization of V. This is relevant in the sense that when the realization is V=2 there is no adverse selection problem. However the type of information sets faced by the p-insider implies the presence of the spread even in the absence of asymmetric information under V=2.

I summarize this result in the following proposition.

Proposition 9. *If condition (*) is true then there is an equilibrium where the quotes are given by:*

$$A^* = q_1 + 2q_2$$

$$B^* = \frac{q_1\,[\rho\,(1-\varepsilon) + (1-\rho)] + 2q_2\rho\,(1-\varepsilon)}{q_1\,[\rho\,(1-\varepsilon) + (1-\rho)] + q_2\rho\,(1-\varepsilon)}$$

In this equilibrium there is only a speculative sell if V=1, but there is no speculative buy.

V. Remarks on the Two Examples

The first example emphasizes the sensitivity of the equilibrium outcomes to the informational assumptions. This may indicate a potential problem implicit in the sequential trade models of market microstructure. In the standard framework (Easley and O'Hara, 1987) the assumption is that the p-insider has perfect information about the true value while the market maker does not have any extra signal. The dealer only infers some of the information through trade. What this first example suggests is that, whenever the standard assumption is dropped, many other possibilities could arise. This depends upon how the information partitions observed by each agent are. The potential problem is that one could suspect that an *anything goes* type of result could arise in this context. This example could show that by using this type of models one could explain any type of behavior by *suitably* changing the information set. Hence, if this were the case, the sequential trade models should be empirically contrasted with the data in order to eliminate this potential problem.

In the second example I get that the speculative trading is just relegated to a sell if the true value is low. There is no purchase of the asset. This second case could be used to start a line of research towards more general models of "pessimistic behavior" in trading. As already mentioned, this type of information sets (which are non partitional) may be interpreted as a simple construction reflecting more complicated behavior such as pessimistic or optimistic. Although I do not plan to cover this last case, it is natural to think about "optimistic" behavior from the part of the p-insider if the information sets look like the set $\{\{1, 2\} \{2\}\}$. This implies that, even though the true realization is $V=1$, the p-insider still considers as a possible value the high one.

However, there is a strong problem with this interpretation. In the second example, whenever the true value is $V=1$, the p-insider still considers as possible both $V=1$ and $V=2$. However the priors do *not* change when the true realization is $V=1$. Consider the story told at the beginning of the Section III. Under this interpretation the p-insider receives some "secret" information from some person about the performance of the firm. Whenever this performance is good, the person informs the p-insider. Since this is "pessimistic" she still considers possible both realizations of the firm. However, the priors of the p-insider are not changed even though this person gives to the p-insider good news about the performance of the firm. This is shown by the fact that, whenever the p-insider calculates the conditional expected value when $V=2$ she uses the original priors. This may not be very intuitive. One could argue that, despite the fact that a trader may be pessimistic, she should use the information at least to update the priors in some way. Instead, not only the p-insider still believes possible that the bad outcome is possible even though the true value is high, but also she still believes that it is possible *under the same priors* as before the realization of the high value. This represents a problem. Nevertheless I still believe that it may constitute a *first* attempt of modelling boundedly rational traders and also pessimistic behavior.[6]

VI. Concluding Remarks and Future Research

The main conclusion is that the assumption about the information availability for each trader is not trivial at all, whenever one has to use the sequential trade framework in market microstructure problems. The first example shows clearly this, by changing the information partitions available to each of the traders. The trade pattern as well as the bid-ask spread are clearly different across different information assumptions. The second example emphasizes this. In addition this second case gives a first attempt of modelling boundedly rational behavior using sequential trade models by adapting *the non-partitional information* device used for the first time by Geanakoplos (1989).

There is an important remark about assumption 1. As the reader may have already noticed, this assumption is crucial for most of the results. It would be important to explore how the results are changed whenever assumption 1 is not true. In particular, assuming that under indifference the p-insider trades may lead

to very different equilibrium outcomes. I leave this as an open question. The important point in the paper is that some type of behavioral supposition is needed due to the risk-neutrality assumption.

This paper offers a good set of possible extensions. On the theoretical side these cases must be generalized to include multiperiod economies, as well as the possibility of not having any realization at all. This is important to see how the results given by Easley and O'Hara among others about the characterization of the quotes processes are changed for different information partitions. Another interesting point to study is the possibility of having revelation of information through time. In a very different framework, Kawamura (1997) showed some examples of finitely repeated games without partitional information sets where the players were able to learn the true realized state of nature after the first round of play. Ideas like this may be important to see if the problem of *non-robustness* of the equilibria is just a problem of considering only one period or if it is a more relevant issue.

On the empirical side it is obvious that a multiperiod version of this could be tested in the same way as the standard models (see, for example, Easley, Kiefer and O'Hara, 1997). Moreover it would be possible to test different information assumptions by solving for the equilibrium under different informational assumptions and then estimating the parameters under the equilibrium. In this way one could defend the sequential trade models since in this way it is possible to infer the information available to each type of trader by testing the equilibrium under different information partitions using econometric techniques. There is still an issue on learning in multi-period extensions. In the literature initiated in Easley and O'Hara (1987), price processes follow a Martingale with respect to the dealer's information. This implies a semi-efficient market as it is discussed in O'Hara (1995) (see chapter 3 for details). The assumption on Bayesian Learning also implies that prices converge to the true value. In the case of my paper it is possible that quotes could also follow a Martingale, obtaining thus convergence. In such a situation one may ask what the role of noise traders is. However this question should also be imposed on the standard model. The answer is that efficient markets in its strong form can never arise here, due to the presence of liquidity traders. A related question is whether the non-robustness results in Section III remain in the long run with the Bayesian Learning approach. Still some other forms of learning could be considered. Adaptive learning is a possibility. In particular Easley and Rustichinni (1999) provide a decision-theoretical foundation of adaptive learning for individual choices. In the context of games Marimon and McGrattan (1995) provide complementary characterizations.

APPENDIX

Proof of Proposition 3

Suppose first that the true realization is V=1. In this case the p-insider knows perfectly the value due to the information that is given to her. Therefore $Z(1)=1$ under this assumption. By the usual formula for the ask and the bid (similar to the formula given in Section II, subsection 2.2) both are strictly in the interval (1, 3). Therefore it is true that $A(1) > Z^{(1)}$ and $B(1) > Z^{(1)}$. Then the following is true:

$$P\,[sell \mid V=1] = \rho\,(1-\varepsilon) + 1 - \rho$$
$$P\,[buy \mid V=1] = \rho\varepsilon$$

On the other hand $V = 4$ implies that $A(4) = B(4) = 4$. Since the bid and the ask are both B measurable, then the p-insider can infer the true value whenever V=4 since the partition faced by the p-insider (ex-post) is constituted by the intersection of the sets of the original p-insider's partition and the elements of the partition generated by the bid and the ask, which is just B. Therefore $Z^{(4)} = 4$ and by our no-trade assumption there is no incentive to trade.

Suppose now that $V = 2$ or $V = 3$. In this case $Z^{(\omega)} = [2q_2+3q_3] / [q_2+q_3]$ for $\omega = 2, 3$. Suppose that the market maker knows that with the bid and the ask corresponding to these two values the p-insider does not sell but she buys the security. Then I have the following conditional probabilities.

$$P\,[buy \mid V=\omega] = \rho\varepsilon + 1 - \rho$$
$$P\,[sell \mid V=\omega] = \rho\,(1-\varepsilon)$$
$$\omega = 2, 3$$

With this information I can calculate the bid and the ask for $\omega = 1, 2, 3$. They are given by the conditional expectations in the usual fashion. After some tedious calculations I get that

$$A(\omega) = \frac{q_1\rho\varepsilon + [\rho\varepsilon + (1-\rho)]\,[2q_2 + 3q_3]}{q_1\rho\varepsilon + [\rho\varepsilon + (1-\rho)]\,[q_2 + q_3]}$$

$$B(\omega) = \frac{q_1\,[\rho\,(1-\varepsilon) + 1 - \rho] + \rho\,(1-\varepsilon)\,[2q_2 + 3q_3]}{q_1\,[\rho\,(1-\varepsilon) + 1 - \rho] + \rho\,(1-\varepsilon)\,[q_2 + q_3]}$$

If the guesses are true, then it must be the case that for ω =2, 3, $Z^{(\omega)} > A^{(\omega)}$ and $Z^{(\omega)} > B^{(\omega)}$. Note that by our assumptions

$$q_1\rho\varepsilon(q_2+2q_3) > 0$$

This implies

$$q_1\rho\varepsilon\,(q_2+2q_3) + [\rho\varepsilon+1-\rho]\,[2q_2+3q_3]\,[q_2+q_3]$$
$$> [\rho\varepsilon+1-\rho]\,[2q_2+3q_3]\,[q_2+q_3]$$

Manipulating this:

$$\frac{[2q_2 + 3q_3]}{[q_2 + q_3]} > \frac{q_1\rho\varepsilon + [\rho\varepsilon + 1 - \rho]\,[2q_2 + 3q_3]}{q_1\rho\varepsilon + [\rho\varepsilon + 1 - \rho]\,[q_2 + q_3]}$$

This implies that $Z^{(j)} > A(j)$ for $j = 2, 3$. Therefore the p-insider will buy the asset, confirming the "guess" I made.

For the bid the proof is similar. By the assumptions it is true that:

$$q_1\,[\rho\,(1-\varepsilon) + 1 - \rho]\,[q_2 + 2q_3] > 0$$

which implies

$$q_1[\rho\,(1-\varepsilon) + 1 - \rho]\,[q_2 + 2q_3] + \rho\,(1-\varepsilon)\,(2q_2 + 3q_3)\,(q_2 + q_3)$$
$$> \rho\,(1-\varepsilon)\,(2q_2 + 3q_3)\,(q_2 + q_3)$$

Then

$$\frac{[2q_2 + 3q_3]}{[q_2 + q_3]} > \frac{[q_1[\rho\,(1-\varepsilon) + 1 - \rho] + \rho\,(1-\varepsilon)\,[2q_2 + 3q_3]}{[q_1[\rho\,(1-\varepsilon) + 1 - \rho] + \rho\,(1-\varepsilon)\,[q_2 + q_3]}$$

This implies $Z^{(\omega)} > B(\omega)$ for $\omega = 2, 3$. Then the p-insider will not sell under these two realizations. This completes the proof.

Proof of Proposition 4

First, under V=1 the bid and the ask is equal to 1. By measurability again the information sets faced actually by the p-insider after knowing the partition induced by the bid and the ask plus the original p-insider's partition imply that the p-insider is able to infer correctly the state of the world too. Therefore $Z^{(1)}=1$ and then there is no trade under this realization. On the other hand when V=4 the p-insider directly infers this correctly from her original partition. Then the p-insider has $Z^{(4)}=4$. By the ask and the bid formula I see that both $A(4)$ and $B(4)$ are in (2, 4). Therefore $A(4) < Z^{(4)}$ and $B(4) < Z^{(4)}$. This implies that the p-insider will *buy* and *not sell* whenever V=4 is true.

When V=2 or V=3 I must make again some guesses about speculative trading which must be fulfilled at equilibrium. Suppose that the dealer knows that the p-insiders do not buy but sell the security whenever V=2 or V=3. Then the following are the conditional probabilities.

$$P\,[buy \mid V=\omega] = \rho\varepsilon$$
$$P\,[sell \mid V=\omega] = \rho\,(1-\varepsilon) + 1 - \rho$$

for $\omega = 2, 3$. If this is the case, then the asks and the bids are given by the formulae in the equilibrium.

$$A(\omega) = \frac{(2q_2 + 3q_2)\ \rho\varepsilon + 4q_4\ [\rho\varepsilon + 1 - \rho]}{(q_2 + q_3)\ \rho\varepsilon + q_4\ [\rho\varepsilon + 1 - \rho]}$$

$$B(\omega) = \frac{(2q_2 + 3q_3)\ [\rho\ (1 - \varepsilon) + 1 - \rho] + 4q_4\rho\ (1 - \varepsilon)}{(q_2 + q_3)\ [\rho\ (1 - \varepsilon) + 1 - \rho] + q_4\rho\ (1 - \varepsilon)}$$

for $\omega = 2, 3, 4$. This gives the trading behavior as assumed before. First, by the statement of the proposition:

$$q_4\ (\rho\varepsilon + 1 - \rho)\ [2q_2 + q_3\ ] > 0$$

Therefore

$$q_4\ (\rho\varepsilon + 1 - \rho)\ [2q_2 + q_3] + (2q_2 + 3q_3)\ (q_2 + q_3)\ \rho\varepsilon$$
$$> (2q_2 + 3q_3)\ (q_2 + q_3)\ \rho\varepsilon$$

Then

$$q_4\ (\rho\varepsilon + 1 - \rho)\ [(4 - 2)\ q_2 + (4 - 3)q_3] + (2q_2 + 3q_3)\ (q_2 + q_3)\ \rho\varepsilon$$
$$> (2q_2 + 3q_3)\ (q_2 + q_3)\ \rho\varepsilon$$

Thus

$$\frac{[4q_4\ (\rho\varepsilon + 1 - \rho) + (2q_2 + 3q_3)\ \rho\varepsilon]}{[q_4\ (\rho\varepsilon + 1 - \rho) + \ (q_2 + q_3)\ \rho\varepsilon]} > \frac{(2q_2 + 3q_3)}{(q_2 + q_3)}$$

This implies $A\ (\omega) > Z^{(\omega)}$ for $\omega = 2, 3$. Then the p-insider will *not* buy the security. On the other hand

$$q_4\rho\ (1 - \varepsilon)\ [2q_2 + q_3] > 0$$

Therefore

$$q_4\rho\ (1 - \varepsilon)\ [2q_2 + q_3] + (2q_2 + 3q_3)\ (q_2 + q_3)\ (\rho\ (1 - \varepsilon) + 1 - \rho)$$
$$> (2q_2 + 3q_3)\ (q_2 + q_3)\ (\rho\ (1 - \varepsilon) + 1 - \rho)$$

and

$$\frac{[4q_4\rho\ (1 - \varepsilon) + (2q_2 + 3q_3)\ (\rho(1 - \varepsilon) + 1 - \rho)]}{[q_4\rho\ (1 - \varepsilon) + (q_2 + q_3)\ (\rho(1 - \varepsilon) + 1 - \rho)]} > \frac{(2q_2 + 3q_3)}{(q_2 + q_3)}$$

This implies that $B\ (\omega) > Z^{(\omega)}$. This says that the p-insider will sell the security under $\omega = 2, 3$. This ends the proof of the proposition.

Proof of Lemma 6

By definition of equilibrium the p-insider compares bids and asks with the following expression.

$$Z^{(\omega)} = E\ [V \mid H,\ A,\ B]\ (\omega)$$

The p-insider conditions the expected value to the observed set H in her own partition and the information about the bid and the ask distributions. Since the information partitions are common knowledge (although not the actual realized value) then the p-insider knows the distribution of the equilibrium bid and ask across different states. On the other hand it is also common knowledge that the equilibrium quotes must be measurable with respect to the market maker's partition. Therefore the following is true:

$$A(\omega) = A(\omega')\ \textit{if}\ \omega,\ \omega' \in D_i,\ \textit{same}\ i$$
$$B(\omega) = B(\omega')\ \textit{if}\ \omega,\ \omega' \in D_i,\ \textit{same}\ i$$

$$A(\omega) \neq A(\omega')\ \textit{if}\ \omega \in D_i,\ \omega' \in D_j,\ i \neq j$$
$$B(\omega) \neq B(\omega')\ \textit{if}\ \omega \in D_i,\ \omega' \in D_j,\ i \neq j$$

This also generates a partition of the whole state space ΩI, which is of course the same partition as the one observed by the specialist. In other words, the partition of the dealer and the partition induced by equilibrium bid and asks must be the same. (Again this *must* be so because of the measurability condition). Since this partition is then also observed by the p-insider in equilibrium, then the p-insider observes both her own partition and the dealer's partition, as claimed in the lemma.

Proof of Proposition 5

Suppose first that $M(\omega) = \{\omega\}$. Then by the lemma:

$$\begin{aligned} Z^{(\omega)} &= E\ [V \mid I_j,\ A,\ B] \\ &= \frac{q_\omega}{q_\omega}\ \omega \\ &= \omega \end{aligned}$$

Suppose now that $M(\omega)$ is according to the second possibility. That means that it is not a singleton and the information sets defining $M(\omega)$ are such that the element of the p-insider's partition containing ω is included in the information set of the dealer. This trivially implies that $M(\omega) = D_i(\omega)$, some i. Then the value of $Z^{(x)}$ for any x in $M(\omega)$ is given by the following expression: (dropping the state ω from $M(\omega)$)

$$Z = \frac{\sum_{i:\omega_i \in M} \omega_i q_i}{\sum_{i:\omega_i \in M} q_i}$$

Suppose now that the dealer thinks that with the bid and the ask she will post the p-insider will not buy the asset under any x in M. Therefore

$$P\,[buy \mid V = x\,] = \rho\varepsilon$$

Therefore

$$P[V = x \mid buy] = \frac{q_x}{\sum_{i:\omega_i \in M} q_i}$$

for any x in M. Then the ask is equal to

$$A(x) = \frac{\sum_{i:\omega_i \in M} \omega_i q_i}{\sum_{i:\omega_i \in M} q_i}$$

which is equal to Z. By assumption 1, the p-insider does not buy the asset, confirming the belief of the dealer. Suppose on the other hand that the market maker thinks that the p-insider does not sell under any of the states in M. Therefore for any x in M

$$P\,[sell \mid V = x] = \rho\,(1 - \varepsilon)$$

Then

$$P[V = x \mid sell] = \frac{q_x}{\sum_{i:\omega_i \in M} q_i}$$

for any x in M. Then the bid is equal to

$$B(x) = \frac{\sum_{i:\omega_i \in M} \omega_i q_i}{\sum_{i:\omega_i \in M} q_i}$$

which is equal to Z. By assumption 1, the p-insider does not sell the asset, confirming the belief of the dealer. Moreover the bid and the ask coincide for any x in M. Then there is no bid-ask spread, as claimed.

Proof of Proposition 7

Suppose that the market maker believes that under the ask and the bid she posts the p-insider buys the asset under any x in $M(\omega)$. Therefore

$$P\,[buy \mid V = x] = \rho\varepsilon + (1 - \rho)$$

Note that by assumption, for any y in D_i and not in $M(\omega)$ I have

$$P\,[buy \mid V = y] = \rho\varepsilon$$

Therefore we have the following conditional probabilities

$$P\,[V = x \mid buy] = \frac{q_x\left[\rho\varepsilon + 1 - \rho\right]}{\left[\sum_{\substack{i:\omega_i \in D_i \\ \omega_i \notin M}} q_i\right]\rho\varepsilon + \left[\sum_{j:\omega_j \in M} q_j\right]\left[\rho\varepsilon + 1 - \rho\right]}$$

$$x \in M$$

$$P\,[V = y \mid sell] = \frac{q_y \rho\varepsilon}{\left[\sum_{\substack{i:\omega_i \in D \\ \omega_i \notin M}} q_i\right]\rho\varepsilon + \left[\sum_{j:\omega_j \in M} q_j\right]\left[\rho\varepsilon + 1 - \rho\right]}$$

$$y \in D_i,\ y \notin M$$

This gives the following ask:

$$A = \frac{\left[\sum_{\substack{i:\omega_i \in D \\ \omega_i \notin M}} q_i\omega_i\right]\rho\varepsilon + \left[\sum_{j:\omega_j \in M} q_j\omega_j\right][\rho\varepsilon + 1 - \rho]}{\left[\sum_{\substack{i:\omega_i \in D \\ \omega_i \notin M}} q_i\right]\rho\varepsilon + \left[\sum_{j:\omega_j \in M} q_j\right][\rho\varepsilon + 1 - \rho]}$$

On the other hand, we know that

$$Z^{(x)} = \frac{\sum_{j:\omega_j \in M} q_j\omega_j}{\sum_{j:\omega_j \in M} q_j}$$

for any x in M. This is true by Lemma 2.6. Then the dealer's belief is confirmed if $Z^{(x)} > A(x)$ for all x in $M(\omega)$. If the condition stated in the proposition is true then we clearly have

$$\left[\sum_{\substack{i:\omega_i \in D_i \\ \omega_i \notin I_j}} q_i\right]\left[\sum_{j:\omega_j \in M} q_j\omega_j\right] - \left[\sum_{\substack{i:\omega_i \in D_i \\ \omega_i \notin I_j}} q_i\omega_i\right]\left[\sum_{j:\omega_j \in M} q_j\right] > 0$$

This gives:

$$\rho\varepsilon\left\{\left[\sum_{\substack{i:\omega_i \in D_i \\ \omega_i \notin I_j}} q_i\right]\left[\sum_{j:\omega_j \in M} q_j\omega_j\right] - \left[\sum_{\substack{i:\omega_i \in D_i \\ \omega_i \notin I_j}} q_i\omega_i\right]\left[\sum_{j:\omega_j \in M} q_j\right]\right\} > 0$$

Therefore:

$$\frac{\left[\sum_{j:\omega_j \in M} q_j\omega_j\right]}{\left[\sum_{j:\omega_j \in M} q_j\right]} > \frac{\left\{\rho\varepsilon\left[\sum_{\substack{i:\omega_i \in D_i \\ \omega_i \notin I_j}} q_i\omega_i\right] + [\rho\varepsilon + 1 - \rho]\left[\sum_{j:\omega_j \in M} q_j\omega_j\right]\right\}}{\left\{\rho\varepsilon\left[\sum_{\substack{i:\omega_i \in D_i \\ \omega_i \notin I_j}} q_i\right] + [\rho\varepsilon + 1 - \rho]\left[\sum_{j:\omega_j \in M} q_j\right]\right\}}$$

This is just $Z^{(\omega)} > A(\omega)$ for all ω in M. Then condition (**) is sufficient to have the p-insider buying the security under any of the realizations in M. On the other hand, let us suppose that the market maker believes that the p-insider does not sell the security under any realization ω in M. This implies that for any ω in M.

$$P\,[sell \mid V = \omega] = \rho\,(1 - \varepsilon)$$

We have two subcases. Suppose that for all ω in D_i and not in M we also have

$$P\,[sell \mid V = \omega] = \rho\,(1 - \varepsilon)$$

Then the bid would be

$$B' = \frac{\sum_{i:\omega_i \in M \cup D_i} q_i\omega_i}{\sum_{i:\omega_i \in M \cup D_i} q_i}$$

Note that the value $Z^{(\omega)}$ is

$$Z^{(\omega)} = \frac{\sum_{i:\omega_i \in M} q_i\omega_i}{\sum_{i:\omega_i \in M} q_i}$$

for any ω in M. Therefore we must have (in this subcase) that $Z^{(\omega)} > B(\omega)$. Again, condition (**) implies

$$\left[\sum_{\substack{i:\omega_i \in D_i \\ \omega_i \notin I_j}} q_i\right]\left[\sum_{j:\omega_j \in M} q_j\omega_j\right] - \left[\sum_{\substack{i:\omega_i \in D_i \\ \omega_i \notin I_j}} q_i\omega_i\right]\left[\sum_{j:\omega_j \in M} q_j\right] > 0$$

This implies (after tedious algebra):

$$\left[\sum_{j:\omega_j \in M} q_j\omega_j\right]\left\{\sum_{i:\omega_i \in M \cup D_i} q_i\right\} > \left[\sum_{j:\omega_j \in M} q_j\right]\left\{\sum_{i:\omega_i \in M \cup D_i} q_i\omega_i\right\}$$

and then

$$\frac{\left[\sum_{j:\omega_j \in M} q_j\omega_j\right]}{\left[\sum_{j:\omega_j \in M} q_j\right]} > \frac{\left\{\sum_{i:\omega_i \in M\cup D_i} q_i\omega_i\right\}}{\left\{\sum_{i:\omega_i \in M\cup D_i} q_i\right\}}$$

which implies that $Z^{(\omega)} > B(\omega)$. Therefore the p-insider sells under the case in which the p-insider does not sell for realizations in $D_i \setminus M$.

Suppose now that the p-insider sells the security for realizations in $D_i \setminus M$. Therefore we have

$$P\ [sell \mid V{=}x] = \rho\,(1-\varepsilon)$$

for x in M and for all y in D_i and not in M we have

$$P\ [sell \mid V{=}y] = \rho\,(1-\varepsilon) + 1 - \rho$$

Then the bid is

$$B = \frac{[\rho\,(1-\varepsilon)+1-\rho]\left[\sum_{\substack{i:\omega_i \in D_i \\ \omega_i \notin I_j}} q_i\omega_i\right] + \rho\,(1-\varepsilon)\left[\sum_{j:\omega_j \in M} q_j\omega_j\right]}{[\rho\,(1-\varepsilon)+1-\rho]\left[\sum_{\substack{i:\omega_i \in D_i \\ \omega_i \notin I_j}} q_i\right] + \rho\,(1-\varepsilon)\left[\sum_{j:\omega_j \in M} q_j\right]}$$

The value for Z is the same as before. Therefore we must also have that $B < Z$ to confirm the dealer's guess. By condition (**) we get again

$$\left[\sum_{\substack{i:\omega_i \in D_i \\ \omega_i \notin I_j}} q_i\right]\left[\sum_{j:\omega_j \in M} q_j\omega_j\right] - \left[\sum_{\substack{i:\omega_i \in D_i \\ \omega_i \notin I_j}} q_i\omega_i\right]\left[\sum_{j:\omega_j \in M} q_j\right] > 0$$

This gives:

$$[\rho\,(1-\varepsilon)+1-\rho]\left\{\left[\sum_{\substack{i:\omega_i \in D_i \\ \omega_i \notin I_j}} q_i\right]\left[\sum_{j:\omega_j \in M} q_j\omega_j\right] - \left[\sum_{\substack{i:\omega_i \in D_i \\ \omega_i \notin I_j}} q_i\omega_i\right]\left[\sum_{j:\omega_j \in M} q_j\right]\right\} > 0$$

After some algebra I get

$$\frac{\left[\sum_{j:\omega_j \in M} q_j\omega_j\right]}{\left[\sum_{j:\omega_j \in M} q_j\right]} > \frac{[\rho\,(1-\varepsilon)+1-\rho]\left[\sum_{\substack{i:\omega_i \in D_i \\ \omega_i \notin I_j}} q_i\omega_i\right] + \rho\varepsilon\left[\sum_{j:\omega_j \in M} q_j\omega_j\right]}{[\rho\,(1-\varepsilon)+1-\rho]\left[\sum_{\substack{i:\omega_i \in D_i \\ \omega_i \notin I_j}} q_i\right] + \rho\varepsilon\left[\sum_{j:\omega_j \in M} q_j\right]}$$

This implies that the bid is less than the conditional expected value of the asset for the p-insider under any s in M. Then the p-insider does not sell the security as claimed.

Notes

1 See O'Hara (1995) for a survey, specially chapters 2 and 3.

2 As long as I am concerned, this is the first paper that explores this possibility even by means of examples.

3 Think about an investor who gets secret information about the firm through a *contact*. If the investor is pessimistic enough then he will tend to believe the contact only when this gives to the p-insider bad news. But when the contact has good news the p-insider will tend to think that the bad situation is still possible.

4 This is because $B(1)$ is strictly greater than 1 according to our assumptions.

5 I could say that the p-insider is "pessimistic" in the sense that even though she receives a good signal she still believes that the firm could do well or bad.

6 Other issues can arise within this framework. The approach taken by Geanakoplos has been criticized by authors like Modica and Rustichini (1994). They argue that this type of non-partitional information structures violates the axiom of *symmetry* in modelling unawareness. In particular, they show that Geanakoplos's information sets imply not only that the agent is not aware that a certain event happens, but also that she is aware that the event does not happen. From an epistemic point of view this is not correct. However, my purpose in this paper is not to include awareness in a serious way. Rather it focuses on the effect that a more informed by boundedly rational p-insider has on the equilibrium quotes. I also leave the problem of awareness as a possible future topic of research.

References

EASLEY, D. and M. O'HARA (1987). "Price, Trade Size and Information in Securities Markets", *Journal of Financial Economics* 19 (1), pp. 69-90.

EASLEY, D. and M. O'HARA (1992). "Time and the Process of Security Price Adjustment." *Journal of Finance* 47 (2), pp. 576-605.

EASLEY, D., N. KIEFER and M. O'HARA (1997). "One Day in the Life of a Very Common Stock", *Review of Financial Studies* 10 (5), pp. 805-835.

EASLEY, D. and A. RUSTICHINI (1999). "Choice without Beliefs", *Econometrica* 67, 1157-84.

FOSTER, D. and S. VISWANATHAN (1996)."Strategic Trading When Agents Forecast the Forecasts of Others", *The Journal of Finance* 60 (4), pp. 1437-1478.

GEANAKOPLOS, J. (1989). "Game Theory Without Partitions, and Applications to Speculation and Consensus", Discussion Paper Nº 914, Cowles Foundation.

GLOSTEN, L. and P. MILGROM (1985). "Bid, Ask and Transaction Prices in a Specialist Market With Heterogeneously Informed Traders", *Journal of Financial Economics* 14 (1), pp. 71-100.