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YET ANOTHER PROOF OF THE EXISTENCE OF A COMPETITIVE EQUILIBRIUM*

ANTONIO D'AGATA**
University of Catania

Abstract

This note provides a new proof of the existence of a walrasian equilibrium in pure exchange economies under quite standard and general assumptions. The proof still employs a fixed point argument, however, it is shorter and simpler than those currently available for the case considered.

I. Introduction

There are several proofs concerning the existence of a walrasian equilibrium in pure exchange economies or in economies with production (see, for example, Arrow and Debreu, 1954; McKenzie, 1954; Gale, 1955; Nikaido, 1956; Debreu, 1959; Arrow and Hahn, 1971; Dierker, 1974). Generally, they are not simple proofs, although several elementary proofs (i.e. not employing fixed point arguments) have been provided under the restrictive assumption that goods are (weak) gross substitutes (see e.g. Nikaido, 1964; Kuga, 1965; Greenberg, 1977; Hildebrand and Kirman, 1988).

In this note, a new proof of the existence of a walrasian equilibrium in pure exchange economies is provided under quite standard and general assumptions. The idea underlying the proof is that the equilibrium price vector is the fixed point of a correspondence whose values are the maximizers of a linear function

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** Faculty of Political Sciences. Email: adagata@lex.unict.it

defined by the excess demand vector. The proof still employs a fixed point argument, however, it is shorter and simpler than those referred above and than those available on textbooks (see e.g. Mas-Colell, Whinston and Green, 1995, p. 585 ff.) for the general case considered here; moreover, it is more complete than simple proofs available on textbooks which, for the sake of simplicity, do not deal appropriately with the behaviour of the excess demand function at the boundary of the price space (see e.g. Varian, 1992, p. 321).

II. The Proof

Let us consider a pure exchange economy with ℓ goods and n households. Denote by G the index set of goods and by H the index set of households. Household $h \in H$ is defined by the triplet $\left((X_h), \left(\underset{\sim h}{>}\right), (\omega_h)\right)$ which has the usual meaning. Let us adopt the following very standard assumption (see, for example, Mas-Colell, Whinston and Green, 1995, p. 581).

Assumption 1. For each $h \in H$:

(i) $X_h = R^{\ell}_{++}$.

(ii) $\sum_{h \in H} \omega_h \in R^{\ell}_{++}$.

(iii) Preferences are continuous, strongly monotone and strictly convex.

Denote by p the price vector, by $z_h : R^{\ell}_{++} \to R^{\ell}$ the excess demand function of agent $h \in H$, by Δ the unit simplex and set: $\Delta^{(n)} = \left\{p \in \Delta \mid p_i \geq 1/(1+n)\ell, i \in G\right\}$, $I\Delta^{(n)} = \left\{p \in \Delta^{(n)} \mid p_i > 1/(1+n)\ell, i \in G\right\}$, $B\Delta^{(n)} = \left\{p \in \Delta^{(n)} \mid \exists\, i \in G : p_i = 1/(1+n)\ell, i \in G\right\}$, where n = 1, 2, … Symbols $I\Delta$ and $B\Delta$ have an obvious meaning. Finally, the symbol #A denotes the cardinality of the generic set A. The price vector $p^* \in \Delta$ is said a *walrasian equilibrium* if $z(p^*) = \Sigma_{h \in H} z_h(p^*) = 0$. The following result is standard (see, for example, Aliprantis, Brown and Burkinshaw, 1990, Sections 1.3 and 1.4).

Lemma 1. Under Assumption 1 function $z : R^{\ell}_{++} \to R^{\ell}$ satisfies the properties:

(i) It is continuous.

(ii) It is homogeneous of degree zero.

(iii) $z(p) \cdot p = 0$ for every p.

(iv) If $\{p^n\} \to p$, $p \quad 0$, $p_i > 0$ for some $i \in G$, then there is a positive number Z such that $-Z < z_i(p^n) < Z$ for every n.

(v) If $\{p^n\} \to p$, $p \quad 0$, $p_i = 0$ for some $i \in G$, then there exists $j \in G$ such that $z_j(p^n) \to +\infty$.

Homogeneity allows to take the price vectors in the unit simplex Δ. Given a sequence of prices $\{p^n\}$ in Δ, define the set $Z(\{p^n\}) = \{i \in G \mid z_i(p^n) \rightarrow +\infty)$.

Proposition. Under Assumption 1 there exists a walrasian equilibrium.

Proof. For every n = 1, 2, …, define de correspondence $\Phi^{(n)}: \Delta^{(n)} \rightarrow \Delta^{(n)}$ as follows: given $p° \in \Delta^{(n)}$, $\Phi^{(n)}(p°) = \{p \in \Delta^{(n)} \mid z(p°)(p - p°) \quad z(p°)(p' - p°), p' \in \Delta^{(n)}\}$. Clearly, $\Phi^{(n)}$ is upper hemicontinuous and convex valued for every $n > 0$, therefore it has a fixed point $p^{(n)} \in \Phi^{(n)}(p^{(n)})$. Notice that at the fixed point $p^{(n)}$: $0 \quad z(p^{(n)})(p' - p^{(n)})$ for every $p' \in \Delta^{(n)}$. Suppose now for the moment that $p^{(n)} \in I\Delta^{(n)}$ and that $z(p^{(n)}) \quad 0$. Thus, by Lemma 1(iii), there are indices $j, h \in G$ such that $z_j(p^{(n)}) > 0$ and $z_h(p^{(n)}) < 0$. Take p' in such a way that $p_i' = p_i^{(n)} + \delta_i$, where $\delta_i = 0$ for $i \quad h, j$, $\delta_j > 0$ and $\delta_h = -\delta_j$ and where δ_j is such that $p' \in I\Delta^{(n)}$ (the existence of such a δ_j follows from the fact set $I\Delta^{(n)}$ is open). Then, $z(p^{(n)})(p' - p^{(n)}) > 0$, a contradiction. Thus, we have shown that if $p^{(n)} \in I\Delta^{(n)}$, then $z(p^{(n)}) = 0$, that is $p^{(n)}$ is a walrasian equilibrium.

In order to complete the proof we have to show that there exists a number $n^* > 0$ such that $p^{(n^*)} \in I\Delta^{(n^*)}$. Suppose not. Therefore for every $n > 0$, $p^{(n)} \in B\Delta^{(n)}$. By compactness, the sequence of fixed points $\{p^{(n)}\}$ converges to $p^* \in \Delta$; in particular $p^* \in B\Delta$. The last inclusion, together with Lemma 1(v) imply that $Z(\{p^{(n)}\})$ is non-empty. By assumption, for every n, $p^{(n)}$ is a fixed point of $B^{(n)}$, thus, for what has been said above, $z(p^{(n)})(p' - p^{(n)}) = \sum_{i \in Z(\{p^{(n)}\})} z_i(p^{(n)})(p_i' - p_i^{(n)}) + \sum_{j \in G \setminus Z(\{p^{(n)}\})} z_j(p^{(n)})(p_j' - p_j^{(n)}) \quad 0$ for every $p' \in \Delta^{(n)}$ and for every n. Clearly, for every natural number N, $\frac{1}{(1+N)\ell} < \frac{1}{\ell} < \frac{1}{\# Z(\{p^{(n)}\})}$ choose N and a number ε such that $\frac{1}{(1+N)\ell} < \varepsilon < \frac{1}{\# Z(\{p^{(n)}\})}$ and consider the price vector p^ε defined as follows: $p_i^\varepsilon = \varepsilon > 0$ if $i \in Z(\{p^{(n)}\})$, and $p_i^\varepsilon = \frac{1 - \# Z(\{p^{(n)}\})\varepsilon}{\ell - \# Z(\{p^{(n)}\})} 0$ if $i \in G \setminus Z(\{p^{(n)}\})$, where the last inequality follows from the way in which ε has been defined. Clearly, $p^\varepsilon \in \Delta$ for every n; moreover, by the strict positivity of p^ε it follows that the $p^\varepsilon \in I\Delta^{(n)}$ for n beyond some N^*; withouth loss of generality we can assume that $N^* > N$. By the assumption that $p^{(n)} \in B\Delta^{(n)}$ for every n, it follows that for $n > N^* > N$: $p_i^\varepsilon > p_i^{(n)}$ if $i \in Z(\{p^{(n)}\})$. Thus, for $n > N^*$ and n "big enough" one has that $p^\varepsilon \in I\Delta^{(n)}$ and that $z(p^{(n)})(p^\varepsilon - p^{(n)}) = \sum_{i \in Z(\{p^{(n)}\})} z_i(p^{(n)})(p_i^\varepsilon - p_i^{(n)}) + \sum_{j \in G \setminus Z(\{p^{(n)}\})} z_j(p^{(n)})(p_j^\varepsilon - p_j^{(n)}) > 0$, because $(p_i^\varepsilon - p_i^{(n)}) > 0$ and $z_i(p^{(n)}) \rightarrow +\infty$ for $i \in Z(\{p^{(n)}\})$, and because $-Z < z_j(p^{(n)}) < Z$, for every $j \in G \setminus Z(\{p^{(n)}\})$. But this contradicts the fact that $p^{(n)}$ is a fixed point for every n.■

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