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A NOTE ON THE DISCOUNT FOR LACK OF MARKETABILITY*

UNA NOTA SOBRE EL DESCUENTO POR FALTA DE COMERCIALIZACION

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Abstract

In this paper we first demonstrate that the Discount for Lack of Marketability (DLOM) derived from option-based models can be replicated using an at-the-money (ATM) put option approximation. We examine three well-known models used in the industry and show that their DLOM estimates are proportional to asset volatility, offering a consistent framework for comparison. Second, we extend the framework to consider time-variant variance by means of GARCH models. We report that a zero-order approximation of the model provides accurate estimates of DLOM for a given set of parameters. This suggests the use of the average between long-run and current variances.

Keywords: *Marketability, Chaffe model, Longstaff model, Finnerty model, GARCH models.*

JEL Classifications: *C52, C58, D82, G13.*

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Resumen

En este artículo, primero demostramos que el descuento por falta de comercialización derivado de modelos basados en opciones puede replicarse mediante una aproximación. Examinamos tres modelos ampliamente utilizados en la industria y mostramos que sus estimaciones de este descuento son proporcionales a la volatilidad del activo, lo que proporciona un marco consistente para su comparación. En segundo lugar, extendemos el marco que utiliza opciones para considerar una varianza que cambia en el tiempo mediante modelos GARCHs. Mostramos que una aproximación de orden cero del modelo proporciona estimaciones apropiadas del descuento para un conjunto dado de parámetros. Esto sugiere utilizar como insumo el promedio entre la varianza de largo plazo y su valor actual.

Palabras Clave: Comercialización, Modelo de Chaffe, Modelo de Longstaff, Modelo de Finnerty, Modelos GARCHs.

I. INTRODUCTION

Marketability in valuation refers to an asset's ability to be quickly converted into cash with minimal cost and is used for assessing the trading restrictions of certain assets (Vianello, 2022). The Discount for Lack of Marketability (DLOM) accounts for these trading restrictions and is different from liquidity but related in specific cases. A liquid asset is marketable without restrictions, while a non-marketable asset is illiquid with restrictions. However, an illiquid asset doesn't always mean it's non-marketable; it may be converted to cash but with significant loss or delay. Empirical studies have examined restricted stock data and pre-initial public offering (pre-IPO) events. The regulations governing restricted transactions, particularly Rule 144, have evolved over time, reducing the holding periods to 2 years (1972), 1 year (1997), and 6 months (2008), resulting in anticipated lower discounts (Strickland, 2022).[1] Additionally, many of these studies face limitations due to insufficient sample sizes, leading to unreliable statistics and wide confidence intervals (Vianello, 2021). Regarding pre-IPO events, a primary concern is determining an appropriate duration for comparable periods in the analysis; while some studies use a few days, others consider several months.

For institutional investors, the DLOM serves as a measure of illiquidity risk that has direct implications on multi-asset portfolio decisions. It represents the additional compensation that is required when an asset cannot be sold quickly due to its limited

[1] Rule 144 regulates the transaction of restricted securities, enhancing transparency and fairness (US-SEC, 2013). The rule defines five conditions to validate the transaction, including that there is adequate public information available to investors about the firm, and the holding period that we reference here.

liquidity. DLOM helps investors estimate the return they should demand for holding illiquid investments. These estimates inform asset allocation and portfolio optimization models by allowing comparisons between liquid and illiquid assets, ensuring portfolios are balanced according to investors' risk tolerance and compensated for reduced liquidity. This concept is used across various asset classes, including private equity, real estate, infrastructure, and private credit. Beyond that, DLOM is also employed in stress testing frameworks that simulate liquidity constraints and for regulatory and audit requirements related to fair value reporting.

Analytical works on option-pricing models are founded on the assumption that the price of restricted stock (i.e., share price) can be equated with the price of an unrestricted stock. This approach is utilized to value the option of a specific "protective put." The concept was introduced by Chaffe (1993), who proposed setting the strike price of the option equal to the initial share price. He further suggested that the stock price could be modeled by a Geometric Brownian Motion (GBM), leading to the standard Black-Scholes-Merton (BSM) formula. Considering that the model excludes a dividend rate, if the interest rate is zero, this constitutes an at-the-money (ATM) put, which can be approximated through a Taylor expansion. Thus, we consider the approach presented in Barenbaum *et al.* (2015) in which an ATM put is used to represent Chaffe's model. Longstaff (1995) provides an analytical upper bound for the DLOM, representing a protective put with a strike price fixed at the maximum observed value within the restricted period. This upper bound assumes perfect market timing by the investor, thus reflecting the lost opportunity due to lack of marketability. As the strike price is stochastic, this scenario exemplifies a lookback put option. Assuming a GBM for the stock price, there exists a closed-form expression for the upper bound of the DLOM, based on the density of the maximum of the Brownian Motion. Notably, this expression does not account for the interest rate, and the GBM model does not incorporate a dividend rate. Finally, Ghaidarov (2009) and Finnerty (2012) suggest using the average stock price as the strike price in the protective put, which is known as an Asian put option. Although there is no closed-form solution for these options under the GBM assumption, the authors provide approximations based on the moment generating function. In these studies, DLOM expressions depend on the dividend rate. Strickland (2022) highlights issues with the dividend rate and proposes a practical solution involving repeated application of the formula. Accordingly, we consider a model with zero-dividend rate to ensure comparability across all models.

In this paper, we first demonstrate that previous models can be replicated using first-order Taylor approximations. Consequently, the DLOM for each model corresponds to a fraction of the relevant volatility level as follows: (i) 23% for Finnerty & Ghaidarov's model, (ii) 40% for Chaffe's model, and (iii) 80% for Longstaff's model. For instance, if the relevant volatility is 30% and the restricted period is one year (360 days), the DLOM's are 6.9%, 12%, and 24%, for the respective models. Second, we extend the Chaffe's model by replacing the GBM assumption with GARCH models. The latter, which account for time-variant volatility, are widely used for modeling financial

assets and are included in standard statistical packages. Duan (1995) suggests using time-series returns to estimate GARCH parameters and simulating option prices by adjusting the measure within the GARCH framework. Under Gaussian disturbances, the change of measure in the GARCH models is straightforward; extensions to non-Gaussian distributions are also possible but these are beyond the scope of this paper. Thus, Monte Carlo simulations are required to compute option prices, which we illustrate using a given calibration of the parameter of the model that is in line with empirical results for US stock indexes. Our results stress that the initial volatility is crucial to compute the DLOM as that is larger when that input is above the long-run volatility (LRV). We also provide an approximation based on average variance, which is obtained by considering zero-order approximations of GARCH and Threshold-GARCH (TGARCH) models. Our empirical exercises validate the use of the approximation for the case of GARCH model, providing a simple proposal for the DLOM which accounts for time-variant volatility; however, the cost of using it is to have a higher estimate for DLOM. For the case of TGARCH model, we note that the zero-order approximation deviates significantly from the simulated figures.

The paper is organized as follows. Section II reviews the current models, providing first-order approximations, which are validated numerically. Section III introduces the GARCH models, their change of measure, the zero-order approximations, and the pricing of ATM put. Also, the section presents and discusses the simulated results for GARCH and TGARCH models. Section IV summarizes the results and makes concluding remarks.

II. A REVIEW OF CURRENT MODELS

In this section we briefly discuss three proposed formulas for the DLOM based on option pricing models: (i) Chaffe (1993), (ii) Longstaff (1995), and (iii) Finnerty (2012)-Ghaidarov (2009).[2] All the authors assume that the unrestricted asset could be continuously traded in a frictional market and that its value dynamics $\left(S_t\right)$ under the risk-neutral measure, follows a Geometric Brownian Motion (GBM): $dS_t = rS_t dt + \sigma S_t dB_t$, where r is the risk-free interest rate, σ is the volatility, and B_t is a standard Wiener process.[3]

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[2] Ghaidarov (2009) provides a discussion of Finnerty (2002), an early paper that was amended and published in 2012. A survey conducted by Business Valuation Resources in 2021 (https://www.bvresources.com/articles/bvwire/the-most-popular-option-model-for-estimating-dlom) concluded that 47% of respondents uses option pricing models to estimate DLOM. The most cited model is Finnerty (57%), follow by Chaffe (27%), and Longstaff (25%). A similar result is obtained in a survey conducted in 2024 (https://www.bvresources.com/articles/bvwire/which-option-based-methods-are-favored-for-dlom).

[3] Barenbaum *et al.* (2015) state that the estimates of DLOM based on put options are overstated as that financial instrument protect from the downside risk but does not eliminate upside gains, thus their proposal is to use an equity collar which requires being long on a put, and short on a call. For simplicity,

II.1. Taylor Approximations

Chaffe (1993) uses the current value as the strike price leading to a formula that depends on the interest rate. Based on Strickland's (2022) discussion, we consider an at-the-money (ATM) put option instead, which is in line with Barenbaum *et al.* (2015) statement (page 68): "The underlying theory for using the cost of a put option to determine a lack of marketability discount is that the at-the-money put would secure the common shareholder's current price by providing protection from downside risk." This implies that the DLOM is $D_F \equiv e^{-rT} E_0^* \left[\left(S_0 e^{rT} - S_T \right)^+ \right] / S_0$, where $E_0^*()$ is the conditional expectation at time-0 using the risk-neutral measure, and $(x)^+ = x$ if x is positive or zero otherwise. The solution for D_F can be expressed in closed form as:

$$D_F = \Phi\left(\frac{\sigma\sqrt{T}}{2}\right) - \Phi\left(-\frac{\sigma\sqrt{T}}{2}\right) = 2\Phi\left(\frac{\sigma\sqrt{T}}{2}\right) - 1, \tag{1}$$

where $\Phi()$ is the cumulative normal distribution function. A linear approximation of (1) is (see Appendix A): $D_F \cong 0.40\sigma\sqrt{T}$.

Table 1 presents a comparison between the exact values obtained from equation (1) and their approximations across various levels of volatility and durations. For instance, at a volatility level of 30%, the DLOM for 20 days is 2.82% based on equation (1) and 2.83% using its approximation. Additionally, for a duration of 2 years and a high volatility level of 50%, the approximation overestimates the discount by only 0.7% (27.6% versus 28.3%). Consequently, the approximation appears to be accurate over a broad spectrum of volatilities and durations. Furthermore, it is observed that the linear approximation displays the "expected shape" for a discount function: increasing over time but at a diminishing rate (Strickland, 2022).

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we consider that both options are ATM, although the authors suggest writing the call ATM and the put at the current value. In our case, the collar has zero cost and therefore the DLOM is only associated with "the financing cost to monetize the asset" (page 71). As the cost of borrowing is higher than the risk-free rate, the DLOM is positive; however, that is an additional information which is not considered in the previous models making non comparable.

TABLE 1

DLOM FOR CHAFFE'S MODEL (FORWARD-STRIKE)
(Percentages)

days	10		20		30		50	
5	0.470	0.471	0.940	0.943	1.410	1.414	2.350	2.357
10	0.665	0.667	1.330	1.333	1.995	2.000	3.324	3.333
20	0.940	0.943	1.880	1.886	2.820	2.828	4.699	4.714
30	1.152	1.155	2.303	2.309	3.454	3.464	5.753	5.774
60	1.629	1.633	3.256	3.266	4.883	4.899	8.129	8.165
90	1.995	2.000	3.988	4.000	5.979	6.000	9.948	10.000
180	2.820	2.828	5.637	5.657	8.447	8.485	14.032	14.142
360	3.988	4.000	7.966	8.000	11.924	12.000	19.741	20.000
720	5.637	5.657	11.246	11.314	16.800	16.971	27.633	28.284

Note: First columns are obtained using equation (1), and second columns (in gray) are based on the approximation.

Longstaff (1995) presents a closed-form expression for DLOM, assuming the investor has perfect market timing ability, and his formula is regarded as an upper-bound for the DLOM. In this case, the put is $D_M \equiv e^{-rT}E_0^*\left[\left(M_T - S_T\right)^+\right]/S_0 = e^{-rT}\left[E_0^*\left(M_T\right) - E_0\left(S_T\right)\right]/S_0 = e^{-rT}E_0^*\left(M_T\right)/S_0 - 1$, where M_T is the payoff at time-T if the asset were sold optimally, which means: $M_T = \max_{\tau} e^{r(T-\tau)}S_\tau$. The solution for D_M can be expressed in closed form as a particular case of a lookback put option:

$$D_M = \left(2 + \frac{\sigma^2 T}{2}\right)\Phi\left(\frac{\sigma\sqrt{T}}{2}\right) + \sigma\sqrt{T}\cdot\phi\left(\frac{\sigma\sqrt{T}}{2}\right) - 1, \tag{2}$$

where $\phi()$ is the normal density function. Note that the formula (2) has been widely referenced in technical textbooks using the exponential function instead of normal density function (Hitchner, 2003; Pratt, 2009; Hitcher *et al.*, 2017).

After some algebra we can write this discount (at the maximum) as a function of the previous one (ATM): $D_M = D_F + \left[0.5\sigma^2 T\Phi\left(0.5\sigma\sqrt{T}\right) + \sigma\sqrt{T}\phi\left(0.5\sigma\sqrt{T}\right)\right]$. Brooks (2014) defines the latter decomposition as plain vanilla put plus residual lookback portion. Note that the first term in the bracket can be ignored as it is of higher order, because is at least of order volatility squared. Given that, a linear approximation of equation (2) is (see Appendix A): $D_M \cong 0.80\sigma\sqrt{T}$.

Table 2 compares the values obtained from equation (2) with those from the linear approximation. For instance, at a volatility level of 30%, the DLOMs for 20 days are 5.77% using equation (2) and 5.66% using the linear approximation. For a term

of 2 years (1800 days) with a high volatility level (50%), equation (2) yields 70%; meanwhile, the approximation suggests a 57% discount.[4]

TABLE 2

DLOM FOR LONGSTAFF'S MODEL (MAXIMUM-STRIKE)
(Percentages)

days	10		20		30		50	
5	0.944	0.943	1.895	1.886	2.852	2.828	4.789	4.714
10	1.337	1.333	2.688	2.667	4.052	4.000	6.825	6.667
20	1.895	1.886	3.817	3.771	5.768	5.657	9.756	9.428
30	2.324	2.309	4.691	4.619	7.100	6.928	12.047	11.547
60	3.299	3.266	6.683	6.532	10.153	9.798	17.357	16.330
90	4.052	4.000	8.232	8.000	12.542	12.000	21.562	20.000
180	5.768	5.657	11.793	11.314	18.082	16.971	31.481	28.284
360	8.232	8.000	16.984	16.000	26.276	24.000	46.559	40.000
720	11.793	11.314	24.643	22.627	38.605	33.941	70.087	56.569

Note: First columns are obtained using equation (2), and second columns (in gray) are based on the approximation.

Finnerty (2012) and Ghaidarov (2009) revisit the use of Chaffe's model by using a put option with the average spot price $\left(A_T\right)$ as the strike price, then we have $D_A \equiv e^{-rT}E_0^*\left[\left(A_T - S_T\right)^+\right]/S_0$. This is referred to as an Asian option, for which there is no closed-form expression in the case of GBM. They argue that Longstaff's model is based on the unrealistic assumption of perfect market timing, leading to an overestimation of DLOM. They suggest an approximation based on the ATM put option, where the proportional volatility used in equation (1) is replaced by an adjusted one $\left(\vartheta\sqrt{T}\right)$. The approximation is compared with Monte Carlo simulations, showing accurate results for a range of volatilities (up to 50%) and terms (up to 5 years). The proposed formula is:

$$D_A = 2\Phi\left(\frac{\vartheta\sqrt{T}}{2}\right) - 1, \tag{3}$$

[4] Strickland (2022) states that a common practice in the industry is to normalize Longstaff's result to ensure is below 100%, which is an event that happens with large volatility and long term. Abbott (2009) and Vianello (2009) argue that this is the correct way to use the model. In this example, the normalization implies a value of 0.57/(1+0.57) which is 36%. Abbott (2023) uses the "option to exchange one asset for another" argument to support the normalization. He assumes that the restricted asset should be used as numeraire, dropping its volatility to zero and obtaining from the Margrabe's equation a call for the first asset (the unrestricted). In this framework, the value of the put can be recovered by computing the normalization.

where $\vartheta^2 T = \Psi_T + \sigma^2 T - 2\ln\left(e^{\sigma^2 T} - 1\right)$ is the adjustment suggested by Finnerty (2012), meanwhile the one presented by Ghaidarov (2009) is $\vartheta^2 T = \Psi_T - 2\ln\left(\sigma^2 T\right)$, where the common component is $\Psi_T = \ln(2) + \ln\left(e^{\sigma^2 T} - \sigma^2 T - 1\right)$.[5]

Like Chaffe's model, we can use the same ATM option approximation: $D_A \cong 0.40\vartheta\sqrt{T}$, and using Taylor approximation we have the following $\vartheta^2 T \cong \sigma^2 T / 3$ for both adjustment terms (see Appendix A), then $D_A \cong 0.23\sigma\sqrt{T}$. This is also in line with Brooks (2014), who numerically found that the ratio ϑ / σ (page 18): "is relative stable around 57 percent." In other words, the 23% factor is consistent with the 57% of 40%.

Table 3 and 4 compare (3) using Finnerty and Ghaidarov adjustments with the linear approximation. As in the previous cases, particularly for Chaffe's model, the proposed approximation is accurate across a range of volatilities and terms.

TABLE 3

DLOM FOR FINNERTY'S MODEL (AVERAGE-STRIKE)
(Percentages)

days	10		20		30		50	
5	0.271	0.271	0.543	0.542	0.814	0.813	1.357	1.355
10	0.384	0.383	0.768	0.767	1.151	1.150	1.918	1.917
20	0.543	0.542	1.086	1.084	1.628	1.626	2.711	2.711
30	0.665	0.664	1.329	1.328	1.993	1.992	3.318	3.320
60	0.940	0.939	1.879	1.878	2.817	2.817	4.683	4.695
90	1.151	1.150	2.301	2.300	3.447	3.450	5.723	5.750
180	1.628	1.626	3.251	3.253	4.865	4.879	8.045	8.132
360	2.301	2.300	4.589	4.600	6.850	6.900	11.238	11.500
720	3.251	3.253	6.464	6.505	9.602	9.758	15.504	16.263

Note: First columns are obtained using equation (3), and second columns (in gray) are based on the approximation.

[5] Note that the DLOM under Finnerty's adjustment is numerically smaller than under Ghaidarov's, but the difference is relatively small (see Appendix A). Strickland (2002) highlights the artificial ceiling of 32.28%, which Brooks (2014) relates this issue to the logarithm of 2, which is included in the common component and is the upper bound of the modified variance.

TABLE 4

DLOM FOR GHAIDAROV'S MODEL (AVERAGE-STRIKE)
(Percentages)

days	10		20		30		50	
5	0.271	0.271	0.543	0.542	0.814	0.813	1.357	1.355
10	0.384	0.383	0.768	0.767	1.152	1.150	1.920	1.917
20	0.543	0.542	1.086	1.084	1.629	1.626	2.716	2.711
30	0.665	0.664	1.330	1.328	1.995	1.992	3.326	3.320
60	0.940	0.939	1.881	1.878	2.822	2.817	4.707	4.695
90	1.152	1.150	2.304	2.300	3.457	3.450	5.768	5.750
180	1.629	1.626	3.259	3.253	4.892	4.879	8.172	8.132
360	2.304	2.300	4.612	4.600	6.927	6.900	11.596	11.500
720	3.259	3.253	6.529	6.505	9.821	9.758	16.510	16.263

Note: First columns are obtained using equation (3), and second columns (in gray) are based on the approximation.

II.2. Binomial Tree

Following Cox, Ross, and Rubinstein (1979) we assume that the value of the asset can be modeled by a binomial-tree with N independent steps. In each step the asset can either increase by a factor $u > 1$ (event H) or decrease by a factor $d < 1$ (event L); the probability of increase is p. The terminal values are $N + 1$, meanwhile the trajectories are 2^N.

The parameters (u, d, and p) are calibrated according with the first two statistical moments of the asset. Here we follow Jarrow and Rudd' (1983) calibration, which implies fixing $p = 0.5$ and adjusting only the factors accordingly. Under the risk-neutral measure these factors are as follows $u = \exp\left(\theta\delta + \sigma\sqrt{\delta}\right)$ and $d = \exp\left(\theta\delta - \sigma\sqrt{\delta}\right)$, where $\delta = T/N$ is the time divided by the number of steps, and $\theta = r - 0.5\sigma^2$ is the risk free interest rate adjusted with the variance.

As an example, we fix T and consider $N = 2$ steps, therefore we have 4 trajectories which we name as: *HH*, *HL*, *LH*, and *LL*. For each of them, and assuming $S_0 = 1$, we have the following (i) terminal values (S_T): u^2, ud, ud, and d^2, (ii) maximum values (M_T): u^2, u, (1 or ud), and 1, and (iii) average values (A_T): $\left(1 + u + u^2\right)/3$, $(1 + u + 1)/3$, $(1 + d + 1)/3$, and $\left(1 + d + d^2\right)/3$, respectively.

For small σ and $r = 0$, we have $\theta = 0$ which implies $d = 1/u$ and it combines Jarrow-Rudd's calibration with the one originally proposed by Cox-Ross-Rubinstein. Under this setting, the relevant values are (i) S_T: u^2, 1, 1, and u^{-2}, (ii) M_T: u^2, u, 1, and 1, and (iii) A_T: $\left(1 + u + u^2\right)/3$, $(1 + u + 1)/3$, $\left(1 + u^{-1} + 1\right)/3$, and $\left(1 + u^{-1} + u^{-2}\right)/3$.

In Table 5 we consider first-order approximations of each value, which were obtained using the following approximation: $u^k \cong 1 + k\sigma\sqrt{\delta}$. Also, different payoffs are computed in line with the models discussed in the text. For example, we note that the schemes proposed in Longstaff or Finnerty provide positive outcomes in 2 out 4 scenarios, however, the former has higher payoffs in each case. An evaluation of the schemes should be performed by considering the expected payoff, in line with the theory of option prices.

TABLE 5

TRAJECTORIES OF THE BINOMIAL TREE

		Values			Payoffs	
	Terminal	Maximum	Average	Chaffe	Longstaff	Finnerty
	S_T	M_T	A_T	$(1-S_T)^+$	$(M_T-S_T)^+$	$(A_T-S_T)^+$
H	$1+2\sigma\sqrt{\delta}$	$1+2\sigma\sqrt{\delta}$	$1+\sigma\sqrt{\delta}$	0	0	0
HL	1	$1+\sigma\sqrt{\delta}$	$1+\frac{1}{3}\sigma\sqrt{\delta}$	0	$\sigma\sqrt{\delta}$	$\frac{1}{3}\sigma\sqrt{\delta}$
LH	1	1	$1-\frac{1}{3}\sigma\sqrt{\delta}$	0	0	0
LL	$1-2\sigma\sqrt{\delta}$	1	$1-\sigma\sqrt{\delta}$	$2\sigma\sqrt{\delta}$	$2\sigma\sqrt{\delta}$	$\sigma\sqrt{\delta}$

In particular, given that each trajectory has a probability of 25%, then the expected payoffs for each scheme are the following (i) Chaffe: $0.5\sigma\sqrt{\delta} \cong 0.35\sigma\sqrt{T}$, (ii) Longstaff: $0.75\sigma\sqrt{\delta} \cong 0.53\sigma\sqrt{T}$, and (iii) Finnerty & Ghaidarov: $(1/3)\sigma\sqrt{\delta} \cong 0.24\sigma\sqrt{T}$. The first and the third results are close to the ones obtained with the GBM model, meanwhile for Longstaff's case the outcome is significantly lower. This is because the binomial-tree imposes a limited number of transactions, associated with the number of steps, then it limits the gains obtained from trading the asset. Indeed, the weight of the relevant volatility does increase with the number of steps —being 65.9%, 69.6%, and 71.3% for 10, 20, and 30 steps, respectively.

II.3. Discussion

Option-based models for DLOM (Chaffe, Longstaff, Finnerty & Ghaidarov) can be articulated in terms of volatility since this parameter is expected to represent the opportunity cost accurately. Additionally, DLOM exhibits an increasing trend over time, albeit at a diminishing rate; in line with what is expected (Strickland, 2022).

Numerical results support the linear approximations of the DLOM, being a fraction of the relevant volatility level as follows: 23% for Finnerty & Ghaidarov's

model, 40% for Chaffe's model, and 80% for Longstaff's model. This is in line with the strike price considered in each protective put.

As a model based on the GBM assumption can be approximated with the Binomial-Tree-Model (BTM), we found that a simple two-step binomial tree delivers similar factors for Chaffe and Finnerty & Ghaidarov models: 35% for the first and 24% for the second; however, the factor for Longstaff's model of 53% falls short.[6] Table 6 summarizes the results of this section.

TABLE 6

CURRENT MODELS AND THEIR APPROXIMATIONS

Strike	Chaffe	Longstaff	Finnerty
	Forward	Maximum	Average
BSM formula	$D_F = 2\Phi(0.5\sigma\sqrt{T}) - 1$	$D_M = D_F + f(\sigma\sqrt{T})$	$D_A = 2\Phi(g(\sigma\sqrt{T})) - 1$
Taylor Approx.	$D_F \cong 0.40\sigma\sqrt{T}$	$D_M \cong 0.80\sigma\sqrt{T}$	$D_A \cong 0.23\sigma\sqrt{T}$
BTM (N=2)	$D_F \cong 0.35\sigma\sqrt{T}$	$D_M \cong 0.53\sigma\sqrt{T}$	$D_A \cong 0.24\sigma\sqrt{T}$

Note: BSM and BTM stand for Black-Scholes-Merton and Binomial-Tree-Model, respectively. As it is discussed in the text the Taylor approximations are first order and the BTM uses only two steps.

III. AN EXTENSION USING GARCH MODELS

All models presented in the previous section are based on the GBM assumption. This is convenient for having closed-form expressions, but it is rejected empirically. In this section, we propose a simple extension of the framework by introducing GARCH and TGARCH models, which are widely used for modeling several financial assets such as stocks, commodity prices, and currencies. Although the introduction of the GARCH/TGARCH can be used to extend all the models presented here, we focused our discussion solely on Chaffe's model, with the strike price fixed at the forward value. This approach enables us to determine the DLOM based on ATM options, whether calls or puts.

III.1. Physical and Risk Neutral Measures

Stocks are usually modeled using time-variant volatility models such as Stochastic Volatility (SV) or GARCHs. For instance, Heston (1993) proposes an SV model that

[6] We note that BTM requires several steps to achieve a similar factor than Longstaff's model. For instance, considering 30 steps, which involves more than one billion trajectories, we have a factor of 71%.

provides an almost closed-form expression for pricing European options; meanwhile Heston and Nandi (2000) propose a discrete-time version of Heston's model which belongs to the GARCH family. Duan (1995) presents a simple procedure using a GARCH(1,1), which is first estimated under the physical measure and then simulated under the risk-neutral measure.[7]

Following Duan (1995) and Duan *et al.* (2006), we assume that under the physical measure the unrestricted asset can be characterized by the following dynamic:

$$\Delta \log S_t = \left(r - \frac{1}{2}h_t\right) + \lambda\sqrt{h_t} + \sqrt{h_t}u_t, \tag{4}$$

where h_t is conditional variance, λ is the risk-premium parameter, and u_t is a standard Gaussian error term.[8]

We assume that the conditional variance is defined: $h_{t+1} = \omega + \left(\alpha u_t^2 + \beta + \gamma d_t u_t^2\right)h_t$, where d_t is a dummy variable that takes the value of one when $u_t > 0$ and zero otherwise, meanwhile, ω, α, and β are positive parameters, and $\alpha + \beta + \gamma/2 < 1$. The latter is to ensure the existence of unconditional variance. After some algebra, we could write the conditional variance process as: $h_{t+1} = \omega + (\alpha + \beta + \gamma/2)h_t + \alpha\left(u_t^2 - 1\right)h_t + \gamma\left(d_t - 1/2\right)u_t^2 h_t$. Given that u_t is a standard Gaussian error term, the unconditional expected values of the last two terms are zero under the physical measure. Finally, this model is known as Threshold-GARCH (TGARCH) and collapses to the standard GARCH when $\gamma = 0$. The parameters of both models can be estimated with a time-series of returns of the unrestricted asset.[9]

The objective of the change of measure is to have a discounted martingale under the risk neutral measure. To achieve that we consider that $v_t = u_t + \lambda$ is a standard Gaussian error term under the risk-neutral measure. Replacing that in (4) and in the conditional variance formula provides a GARCH model that can be used for pricing options using Monte Carlo simulations, which is:

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7 Several authors extend the framework to others GARCH models (e.g., Exponential GARCH), improving the fit of the empirical returns, and combine returns and option prices (or indexes) to estimate the parameters of these models, improving the precision of the estimates (Hao and Zhang, 2013; Kanniainen *et al.*, 2014; Alfaro and Inzunza, 2024).

8 The model can be extended to non-Gaussian disturbances, which provide more precise estimates of the tails of the distribution; however, the procedure to change the measure (from physical to risk-neutral) does allow to have a simple proposal for the DLOM.

9 We note that the expression in parenthesis in (4) represents adjusted mean under the risk-neutral measure, given that the asset is distributed conditionally log-normal. Standard econometric packages do not allow for including the conditional variance; however, an approximation of this model can be achieved by considering the percentual return instead of logarithm one. This implies estimating a TGARCH(1,1) with ARCH-in-mean term using excess returns.

$$\Delta \log S_t = \left(r - \frac{1}{2}h_t\right) + \sqrt{h_t}v_t, \tag{5}$$

where the conditional variance is: $h_{t+1} = \omega + \left[\alpha\left(v_t - \lambda\right)^2 + \beta + \gamma \tilde{d}_t\left(v_t - \lambda\right)^2\right]h_t$ where $\tilde{d}_t$ is a dummy variable that takes the value of one when $v_t > \lambda$ and zero otherwise.

Iterating (5), the cumulated return can be expressed as $\log S_T = \log S_0 + rT + X_T$, where:

$$X_T = -\frac{1}{2}\sum_{t=0}^{T-1} h_{t+1} + \sum_{t=0}^{T-1}\sqrt{h_{t+1}}v_{t+1}. \tag{6}$$

In general, the distribution of X_T is unknown, as conditional variances depend on previous shocks. However, the data generating process is easy to simulate and, in the next section, we will rely on that to obtain the DLOM as the price of put option ATM.

III.2. Zero-order Approximations

For GARCH model $(\gamma = 0)$ we note that if α is zero then (6) collapses to a simple expression: $X_T \cong -0.5\bar{h}T + \left(\bar{h}T\right)^{1/2}v_T$, where v_T is a standard Gaussian term, and $\bar{h}$ is the average variance defined as (see Appendix B):

$$\bar{h} \equiv h_\infty + \frac{1}{T}\left(\frac{1-\rho^T}{1-\rho}\right)\left(h_1 - h_\infty\right). \tag{7}$$

where $h_\infty = \omega/(1-\rho)$ is the long-run variance and $\rho = \alpha\left(1+\lambda^2\right) + \beta$ is the persistence, both under the risk-neutral measure.

In this case, X_T is normally distributed and the BSM can be used as in Chaffe's model. In other words, $D_G \cong 2\Phi\left(0.5h^{-1/2}\sqrt{T}\right) - 1 \cong 0.4h^{-1/2}\sqrt{T}$. This is the "zero order" approximation of our DLOM with GARCH model.[10]

Analogous, for the TGARCH model we need that both α and γ should be zero to get expression above. In that case the average variance continues

[10] Higher order approximations have been proposed in the literature using the Gram-Charlie density function, which requires approximations of the first four cumulants (Duan *et al.*, 1999; Duan et al, 2006: Mazzoni, 2010). In contrast, perturbation methods have been used in the case of SV models, which are defined in continuous time. Lewis (2000) provides an approximation that can be used in both the Heston and the continuous-time GARCH models. Alternatively, the relationship between local volatility and implied volatility could be used as is explained in detail in Mazzoni (2018). Mazzoni (2015) develops an application to GARCH models, by replacing the Gaussian disturbance with a Brownian bridge shock, emulating the strategy most-likely path.

to be a sufficient statistic, but with a different persistence (Duan *et al.*, 2006): $\rho = \alpha\left(1+\lambda^2\right)+\beta+\gamma\left[\left(1+\lambda^2\right)\Phi(\lambda)+\lambda\phi(\lambda)\right]$.

Notably, the use of $\bar{h}^{1/2}$ acknowledges that initial level of volatility could be above/below the long-run volatility (LRV), and weights accordingly with the maturity of the option.

III.3. Pricing an ATM option

Let $X_{(i)T}$ be a simulated path, then we can obtain the empirical approximation for our proposal DLOM using *m* Monte Carlo (MC) simulations, as follows:[11]

$$D_G = \frac{P_0}{S_0} = \frac{1}{m}\sum_{i=1}^{m}\left(\frac{\left|1-e^{X_{(i)T}}\right|}{2}\right). \tag{8}$$

We note that (8) requires a set of parameters of the model (4) and an initial volatility level. For the latter, we follow Duan (1995) by considering 3 scenarios related with LRV and the initial variance (h_1), setting the latter as follows: (i) 80% of LRV $\left(h_1 = 0.64h_\infty\right)$, (ii) 100% of LRV $\left(h_1 = h_\infty\right)$, and (iii) 120% of LRV $\left(h_1 = 1.44h_\infty\right)$.

Regarding calibration of the parameters, Alfaro and Inzunza (2023) estimate both the GARCH and the TGARCH models, using daily data of S&P 500 index returns and the CBOE VIX index for the period January 2007 – December 2022.[12] From their results we have the following empirical facts for the GARCH model: (i) when only returns are used the estimate for α is 15%, meanwhile when returns and VIX index are considered the estimates for α are between 4% and 6%, (ii) the reduction on this parameter is somehow compensated with the increase in β, and at the end the persistence (ρ) is around 98% in the different settings, and (iii) similarly, for the case of λ, the estimate with only return is 9% and the ones with returns and VIX are between 18% and 27%, being broadly in line with previous studies that acknowledge an increase in the risk premium coefficient when the VIX is included in the estimation. In the case of TGARCH, the persistence is also around 98%, meanwhile estimates for λ, is 5% with only return and between 8% and 16% when both returns and VIX are considered.

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[11] We note that by the call-put parity the price of ATM put is equivalent to the price of ATM call, therefore we can express the put option as the average of both options, which implies the use of the absolute value function.

[12] They consider the VIX index in 3 ways: (i) squared following previous related studies (Hao and Zhang, 2013; Kanniainen *et al.*, 2014), (ii) in levels, as it is usually quoted in the market, and (iii) in logs, as it is used in empirical macro-financial studies.

V-Lab NYU provides estimates of GARCH/TGARCH models for several asset classes, estimated using daily data from January 1990 – June 2025. Table 7 shows a sample of US indexes, that exhibit volatilities from 20% (Dow Jones, Nasdaq, and S&P 500) to 30% (Russell 2000). For the case of GARCH model, all indexes have similar estimates for α and β: having averages of 10% and 88%, respectively. For the case of TGARCH model, similarities remain for β being around 89%, but α and γ exhibit some differences across indexes. In both cases (GARCH and TGARCH), persistence estimates are around 98%-99%.

TABLE 7

GARCH ESTIMATES FOR US STOCK INDEXES

	Dow Jones	Nasdaq	S&P 500	S&P MidCap 400	S&P SmallCap 600	Russell Midcap	Russell 2000
	GARCH						
ω	0.0198	0.0247	0.0185	0.0233	0.0197	0.0244	0.0149
	(0.0009)	(0.0012)	(0.0009)	(0.0009)	(0.0009)	(0.0013)	(0.0008)
α	0.1026	0.1001	0.1025	0.1039	0.0981	0.1161	0.0989
	(0.0025)	(0.0021)	(0.0024)	(0.0024)	(0.0021)	(0.0033)	(0.0021)
β	0.8793	0.8875	0.8824	0.8797	0.8914	0.8669	0.8952
	(0.0025)	(0.0021)	(0.0023)	(0.0024)	(0.0021)	(0.0034)	(0.0020)
$\alpha+\beta$	0.982	0.988	0.985	0.984	0.990	0.983	0.994
σ	19.8	26.8	21.0	22.6	26.0	22.7	30.2
	TGARCH						
ω	0.0217	0.0300	0.0211	0.0264	0.0236	0.0265	0.0182
	(0.0010)	(0.0013)	(0.0010)	(0.0011)	(0.0010)	(0.0016)	(0.0008)
α	0.0109	0.0278	0.0050	0.0228	0.0351	0.0109	0.0394
	(0.0023)	(0.0025)	(0.0026)	(0.0020)	(0.0020)	(0.0030)	(0.0021)
β	0.8920	0.8924	0.8968	0.8892	0.8949	0.8851	0.8980
	(0.0020)	(0.0020)	(0.0022)	(0.0019)	(0.0018)	(0.0025)	(0.0017)
γ	0.1545	0.1269	0.1602	0.1378	0.1131	0.1655	0.1068
	(0.0060)	(0.0053)	(0.0055)	(0.0052)	(0.0050)	(0.0066)	(0.0048)
$\alpha+\beta+\gamma/2$	0.980	0.984	0.982	0.981	0.987	0.979	0.991
σ	19.8	25.7	20.5	22.3	25.1	21.2	26.7

Notes: (1) Estimates are obtained from V-Lab NYU, accessed on June 17th, 2025 (https://vlab.stern.nyu.edu/volatility). (2) Sample: January 1, 1990 — June 13, 2025. (3) Standard errors are in parentheses. (4) TGARCH model is defined as GJR-GARCH in V-Lab NYU.

Based on the previous discussion we calibrate our GARCH using the following set of parameters: (i) $\alpha = 0.1$, $\beta = 0.89$, and $\lambda = 0.2$ which implies that $p = 0.994$, and (ii) $1/\omega = 60000/\sigma^2$, where σ takes the values discussed in the previous section (10%, 20%, 30%, and 50%).

Table 8 shows GARCH results for 5000 MC simulations along with the zero-order approximation presented above. We have the following two observations: (i) the initial level of volatility is crucial to determinate DLOM, which will be higher than usual if h_1 is above LRV (lower panel), and (ii) the approximation provides higher values of DLOM than simulated figures, being the operational cost of not conducting the simulations.

In the case of TGARCH, we fix: (i) $\alpha = 0.02$, $\beta = 0.89$, $\gamma = 0.14$, and $\lambda = 0.1$ which implies that ρ=0.993, and (ii) $1/\omega = 60000/\sigma^2$, where σ takes the values discussed in the previous section (10%, 20%, 30%, and 50%). Table 9 shows results for 5000 MC simulations and the approximation. We note that the zero-order approximation provides inaccurate values for DLOM than simulated figures, therefore the MC simulations are preferred. These results are somehow expected as the zero-order approximation for the TGARCH imposes that both parameters (α and γ) should be simultaneously zero.

TABLE 8

DLOM FOR ATM-GARCH MODEL
(Percentages)

days	10		20		30		50	
5	0.373	0.378	0.746	0.757	1.118	1.135	1.863	1.892
10	0.515	0.537	1.030	1.075	1.544	1.612	2.571	2.687
20	0.739	0.766	1.478	1.532	2.217	2.297	3.691	3.829
30	0.899	0.945	1.797	1.890	2.693	2.835	4.480	4.725
60	1.292	1.363	2.580	2.727	3.864	4.090	6.420	6.816
90	1.563	1.698	3.121	3.397	4.673	5.095	7.756	8.492
180	2.283	2.497	4.556	4.994	6.815	7.492	11.272	12.486
360	3.342	3.693	6.626	7.386	9.854	11.079	16.157	18.465
720	4.945	5.419	9.803	10.839	14.556	16.258	23.658	27.096
5	0.465	0.471	0.929	0.943	1.393	1.414	2.321	2.357
10	0.639	0.667	1.277	1.333	1.915	2.000	3.187	3.333
20	0.909	0.943	1.818	1.886	2.725	2.828	4.537	4.714
30	1.097	1.155	2.192	2.309	3.285	3.464	5.462	5.774
60	1.542	1.633	3.077	3.266	4.608	4.899	7.650	8.165
90	1.830	2.000	3.652	4.000	5.465	6.000	9.060	10.000
180	2.561	2.828	5.106	5.657	7.630	8.485	12.592	14.142
360	3.590	4.000	7.109	8.000	10.557	12.000	17.277	20.000
720	5.135	5.657	10.170	11.314	15.089	16.971	24.494	28.284
5	0.556	0.565	1.113	1.129	1.668	1.694	2.779	2.823
10	0.763	0.797	1.526	1.593	2.287	2.390	3.806	3.984
20	1.081	1.122	2.161	2.244	3.240	3.365	5.392	5.609
30	1.299	1.368	2.595	2.736	3.887	4.104	6.458	6.841
60	1.800	1.912	3.592	3.823	5.375	5.735	8.916	9.558
90	2.110	2.316	4.209	4.632	6.294	6.947	10.419	11.579
180	2.861	3.187	5.699	6.374	8.508	9.561	14.006	15.934
360	3.864	4.346	7.643	8.692	11.332	13.037	18.504	21.729
720	5.352	5.934	10.593	11.869	15.704	17.803	25.449	29.672

Note: First columns are obtained from Monte Carlo simulations, and second columns (in gray) are based on the approximation. Upper, center and lower panels consider 80%, 100%, and 120% of LRV as initial volatility.

TABLE 9

DLOM FOR ATM-TGARCH MODEL
(Percentages)

days	10		20		30		50	
5	0.366	0.343	0.732	0.686	1.098	1.029	1.830	1.715
10	0.491	0.488	0.982	0.975	1.473	1.463	2.455	2.438
20	0.675	0.696	1.352	1.392	2.029	2.088	3.385	3.480
30	0.787	0.860	1.575	1.720	2.364	2.580	3.942	4.299
60	1.029	1.245	2.061	2.489	3.094	3.734	5.167	6.223
90	1.180	1.554	2.362	3.108	3.546	4.662	5.918	7.769
180	1.558	2.291	3.120	4.582	4.686	6.873	7.826	11.455
360	2.108	3.386	4.221	6.771	6.337	10.157	10.571	16.928
720	2.926	4.947	5.862	9.894	8.804	14.842	14.683	24.736
5	0.455	0.427	0.911	0.854	1.366	1.281	2.277	2.135
10	0.607	0.604	1.215	1.208	1.822	1.812	3.036	3.020
20	0.826	0.854	1.653	1.708	2.481	2.562	4.140	4.271
30	0.952	1.046	1.904	2.092	2.858	3.138	4.767	5.230
60	1.205	1.479	2.413	2.959	3.624	4.438	6.055	7.397
90	1.348	1.812	2.698	3.624	4.051	5.435	6.760	9.059
180	1.697	2.562	3.398	5.125	5.103	7.687	8.519	12.812
360	2.218	3.624	4.441	7.247	6.668	10.871	11.122	18.118
720	3.005	5.125	6.021	10.249	9.042	15.374	15.073	25.623
5	0.545	0.511	1.090	1.023	1.635	1.534	2.725	2.557
10	0.725	0.721	1.449	1.442	2.174	2.163	3.622	3.606
20	0.978	1.014	1.959	2.029	2.941	3.043	4.909	5.072
30	1.120	1.236	2.242	2.472	3.365	3.708	5.614	6.181
60	1.389	1.723	2.783	3.447	4.181	5.170	6.989	8.617
90	1.528	2.084	3.059	4.168	4.593	6.253	7.662	10.421
180	1.851	2.859	3.707	5.718	5.566	8.577	9.290	14.296
360	2.343	3.895	4.693	7.790	7.048	11.684	11.750	19.474
720	3.098	5.333	6.208	10.667	9.324	16.000	15.541	26.667

Note: First columns are obtained from Monte Carlo simulations, and second columns (in gray) are based on the approximation. Upper, center and lower panels consider 80%, 100%, and 120% of LRV as initial volatility.

III.4. Discussion

As we show in the previous section, a key parameter in the computation of DLOM is the volatility of the unrestricted asset. The use of GARCH models, discussed in this section, provides a simple and reliable framework to consider time-variant volatility. These can be considered as efficient filters for historical data; thus, no additional smoothing procedures are needed.[13]

[13] We thank Gabriel Petre for raising this issue in an early version of the paper.

For assets related to US equity market, the empirical results suggest that the zero-order approximation could be used for the GARCH model instead of the MC simulations, although it provides a higher value for DLOM. Notably the approximation stresses the difference between current volatility level and the long-run value, a state-dependent feature that is not available in the current option-based models for DLOM. In contrast, the zero-order approximation for the TGARCH significantly overstates the DLOM relative to MC simulations. Based on this, we conclude that MC simulations are strongly preferred for the case of the TGARCH model.

The extension of the proposed framework to emerging markets (EMEs) requires estimates of the GARCH models, a step that is relatively easy to implement. Table 10 shows results for GARCH and TGARCH models of several stock market indexes obtained from V-Lab NYU. As in previous case, α is relatively low (GARCH case) and both α and γ are also low (TGARCH case). This suggests that the zero-order approximation could be applied to these markets, however, the level of LRV is significantly higher than the one obtained for US stock market indexes, preventing for concluding that. Moreover, we believe that some caveats must be considered when applying the change of measure for emerging markets (EMEs).[14] We acknowledge that the lack of market depth could generate a relevant wedge between the GARCH parameters derived from the physical measure with those obtained from the risk-neutral one. Even further, the limited availability of option-like instruments could prevent us to correctly assessing the importance of this issue. This represents a significant challenge that we expect to address in future research.

TABLE 10

GARCH ESTIMATES FOR EMERGING MARKET STOCK INDEXES

	Brazil	Chile	Mexico	Greece	Czech Republic	Thailand	Korea
				GARCH			
ω	0.0141	0.0365	0.0244	0.0228	0.0267	0.0242	0.0179
	(0.0010)	(0.0011)	(0.0011)	(0.0028)	(0.0011)	(0.0028)	(0.0008)
α	0.0356	0.1584	0.0945	0.0794	0.1285	0.1025	0.0945
	(0.0057)	(0.0028)	(0.0026)	(0.0058)	(0.0029)	(0.0025)	(0.0020)
β	0.961	0.8198	0.8952	0.9185	0.8566	0.891	0.9013
	(0.0027)	(0.0026)	(0.0024)	(0.0061)	(0.0026)	(0.0021)	(0.0019)
$\alpha+\beta$	0.997	0.978	0.990	0.998	0.985	0.994	0.996
σ	38.6	24.6	29.2	62.5	25.4	36.6	39.2

[14] We thank our referee for suggesting this emerging market issue.

	Brazil	Chile	Mexico	Greece	Czech Republic	Thailand	Korea
				TGARCH			
ω	0.0117	0.0352	0.0271	0.0245	0.031	0.0267	0.02
	(0.0013)	(0.0011)	(0.0012)	(0.0030)	(0.0011)	(0.0016)	(0.0010)
α	0.0238	0.1109	0.0434	0.0607	0.0861	0.0733	0.0557
	(0.0059)	(0.0037)	(0.0020)	(0.0045)	(0.0036)	(0.0056)	(0.0026)
β	0.9622	0.8295	0.8956	0.9178	0.853	0.8825	0.8995
	(0.0042)	(0.0024)	(0.0020)	(0.0063)	(0.0025)	(0.0024)	(0.0018)
γ	0.0247	0.0814	0.1045	0.0375	0.0834	0.0766	0.079
	(0.0061)	(0.0074)	(0.0064)	(0.0061)	(0.0068)	(0.0165)	(0.0050)
$\alpha+\beta+\gamma/2$	0.998	0.981	0.991	0.997	0.981	0.994	0.995
σ	50.5	25.9	33.4	56.6	24.1	40.4	36.9
	0.9966	0.9782	0.9897	0.9979	0.9851	0.9935	0.9958
	38.6	24.6	29.2	62.5	25.4	36.6	39.2
	0.99835	0.9811	0.99125	0.99725	0.9808	0.9941	0.9947
	50.5	25.9	33.4	56.6	24.1	40.4	36.9

Notes: (1) Estimates are obtained from V-Lab NYU, accessed on June 17th, 2025 (https://vlab.stern.nyu.edu/volatility). (2) Standard errors are in parentheses. (3) TGARCH model is defined as GJR-GARCH in V-Lab NYU.

IV. CONCLUSIONS

In this paper we review option models used to obtain the Discount for Lack of Marketability (DLOM). We provide first-order approximations of the DLOM for popular models, which are based on the Black-Scholes-Merton (BSM) formula. We also validate these results using the binomial tree model (BTM). Besides the convenience of having a simple and comparable formula for these models, we highlight the importance of volatility as the key input for computing DLOM. Both BSM and BTM assume that variance is constant over time, a feature that is strongly rejected empirically. Given that, and following the literature, we propose to use GARCH models for computing DLOM. As there is no closed-form expression for pricing options under this model, we provide zero-order approximations for standard GARCH and Threshold GARCH. We found that the proposed time-variant approximation works relatively well for the GARCH model when parameters calibrated in line with US indexes are used; however, it provides a higher value for DLOM. For the case of TGARCH, we conclude that MC simulations are preferred.

Finally, although GARCH models are easy to implement for any financial market, we caution against a direct application of the standard option-based DLOM as well as the GARCH framework that we proposed here for the case of emerging market economies. We note that the major challenge is to provide a valid procedure to change

the measure from physical to risk-neutral, which is a key element for obtaining DLOM. We leave this issue for future research.

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APPENDIX A

Note that $\phi'(z) = -z\phi(z)$ and $\phi''(z) = (z^2 - 1)\phi(z)$, then $\phi'(0) = 0$ and $\phi''(0) = -\phi(0)$. Thus, we consider second-order Taylor approximation for both the cumulative distribution and additional functions:

$$\Phi(z) \cong \Phi(0) + \Phi'(0)z + \Phi''(0)\frac{z^2}{2} = \frac{1}{2} + \phi(0)z + \phi'(0)\frac{z^2}{2} = \frac{1}{2} + \phi(0)z. \quad \text{(A.1)}$$

$$z\phi(z) \cong 0\phi(0) + [\phi(0) + 0\phi'(0)]z + [2\phi'(0) + 0\phi''(0)]\frac{z^2}{2} = \phi(0)z. \quad \text{(A.2)}$$

We note that both (CDF and additional function) are linear in z, although we consider a second-order approximation.

For Chaffe's model we have $D_F = 2\Phi(z) - 1 \cong 2[0.5 + \phi(0)z] - 1 \cong 2\phi(0)z$. Considering that $z = 0.5\sigma\sqrt{T}$ and $\phi(0) \cong 0.40$, we have $D_F \cong 0.40\sigma\sqrt{T}$. Similarly, for Longstaff's model, we have $D_M = 2[(1 + z^2)\Phi(z) + z\phi(z)] - 1 \cong 2(0.5 + 2\phi(0)z) - 1 \cong 4\phi(0)z$, where we ignore the higher order term (z^2). Taking $z = 0.5\sigma\sqrt{T}$ and $\phi(0) \cong 0.40$, we have $D_M \cong 0.80\sigma\sqrt{T}$.

For Ghaidarov & Finnerty's model, note that $e^x > 1 + x$, then $-2\ln(e^{\sigma^2 T} - 1) < -2\ln(\sigma^2 T)$ therefore the adjusted volatility is higher under Ghaidarov (2009) than under Finnerty (2012), then the former offers a lower DLOM, although the difference is small. To approximate Ψ_T we consider an expansion of the exponential function: $e^{\sigma^2 T} \cong 1 + \sigma^2 T + (1/2)(\sigma^2 T)^2 + (1/6)(\sigma^2 T)^3$, then we have $\Psi_T = \ln(2) + \ln(e^{\sigma^2 T} - \sigma^2 T - 1) \cong 2\ln(\sigma^2 T) + \ln(1 + \sigma^2 T/3)$. For short-term, this implies the following for Ghaidarov's formula: $\vartheta^2 T \cong \ln(1 + \sigma^2 T/3) \cong \sigma^2 T/3$; and this for Finnerty's one $\vartheta^2 T \cong \ln(1 + \sigma^2 T/3) + \sigma^2 T - 2\ln[1 + (1/2)\sigma^2 T + (1/6)(\sigma^2 T)^2] \cong \sigma^2 T/3$. Therefore, the first order approximations are similar, implying that $D_A \cong 0.23\sigma\sqrt{T}$.

APPENDIX B

After some algebra we can write the conditional variance of the GARCH model, under the risk neutral measure, as:[15] $h_{t+1} = \omega + \rho h_t + \alpha h_t e_t$, where $e_t = \left(v_t^2 - 1\right) - 2\lambda v_t$ is the "error term", and $p = \alpha\left(1+\lambda^2\right) + \beta$. The error has zero mean and variance equals to $2 + 4\lambda^2$. Furthermore, if $0 < \rho < 1$ then the unconditional variance exists, and it can be defined as: $h_\infty = \omega/(1-p)$. Using that we have $h_{t+1} = h_\infty(1-p) + \rho h_t + \alpha h_t e_t$, and iterating backward, we have:

$$h_{t+1} = h_\infty + p^t\left(h_1 - h_\infty\right) + \alpha\sum_{i=0}^{t-1} p^i h_{t-i} e_{t-i} = g_t + \alpha\sum_{i=0}^{t-1} p^i h_{t-i} e_{t-i}, \tag{B.1}$$

where $g_t \equiv h_\infty + p^t\left(h_1 - h_\infty\right)$.

When α is zero, the conditional variance is time-variant but deterministic $\left(h_{t+1} \cong g_t\right)$. Thus, under this case, we have:

$$\frac{1}{T}\sum_{t=0}^{T-1} h_{t+1} \cong \frac{1}{T}\sum_{t=0}^{T-1} g_t = h_\infty + \frac{1}{T}\left(\frac{1-p^T}{1-p}\right)\left(h_1 - h_\infty\right) = \bar{h}, \tag{B.2}$$

where $\bar{h}$ is the average variance, and:

$$\sum_{t=0}^{T-1}\sqrt{h_{t+1}}\,v_{t+1} \cong \sum_{t=0}^{T-1}\sqrt{g_t}\,v_{t+1} = \left(\sum_{t=0}^{T-1} g_t\right)^{1/2} v_T, \tag{B.3}$$

where v_T is Gaussian standard. The last step is supported in the fact that g_t is deterministic and therefore the expression is a weighted average of normal disturbances.

Taking previous results, we have: $X_T \cong -0.5\bar{h}T + \left(\bar{h}T\right)^{1/2} v_T$. Then, it has a Gaussian distribution and the BSM formula can be used with $\bar{h}^{1/2}$ as the volatility.

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[15] We note that for the TGARCH model similar arguments can be used. We omit those for brevity.