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GRANDMA'S DRESS, OR WHAT'S NEW FOR OPTIMAL GROWTH

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Abstract:

The recent revival of interest in optimal growth theory justifies the analysis of more flexible descriptions of preferences over time and their implications for optimal growth, in constrat with some of the newer investigations which attribute differing growth paths to tecnological factors. For the case of discrete time, using welfare functions for which the rate of time preference is variable, Beals and Koopmans showed in 1969 that the long run optimal capital path may depend on initial wealth, so that not all of the conclusions of optimal growth theory with a constant rate of time preference hold. Equivalent results for the case of continuous time have been reached by the present author. The analysis by the author of a particular case in which the resulting welfare function can be explicitly represented as an integral, illustrates the qualitative behavior of optimal growth paths. A more thorough analysis shows that a suitable limiting process allows one to define a utility function for continuous time with a variable rate of time preference which cannot be represented in closed form. The present investigation applies such preferences to optimal growth, with results similar to those obtained for the dscrete time case. In particular –when the rate of time preference is allowed to vary– a country may decide not to undertake the effort of economic development when its initial capital

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endowment is below some critical level. It is impossible to obtain such a result with a constant rate of time preference in the case of a simple neoclassical technology.

Introduction

My grandmother used to say that when a dress becomes out of fashion one should just keep it in a trunk for 20 years, time after which one surely would be able to wear it again.

When in June 1992 Robert Barro visited Argentina and gave a thought provoking talk on inter-country comparisons of growth rates due to differing effects of accumulated human capital [Barro and Sala-i-Martin, 1992], I made some comments referring to work done in the 60's, when such differences were attributed to the interaction of preferences with existing resources rather than technological factors. Ile asked me how it fell to see the recent revival of interest in growth theory. My answer was that "it feels wonderful", and cited the foregoing paragraph on grandma's dress. It is the purpose of the present study to present some of the results which provide such an alternative explanation.

For a long time, since the times of the pioneering work by Ramsey [1928], the usual procedure in the field of optimal growth theory –exhaustively examined by Cass [1965] and Koopmans [1965] and applied in a large number of earlier and later studies– consisted in maximing a welfare function represented by a sum or integral of instantaneous utilities of the consumption rates at different times of programs extending to an unlimited planning horizon. The sum was either undiscounted as in Ramsey's case –that author thought any discounting of future generations was nont defensible on ethical grounds– or discounted using a constant rate of time preference. Such a optimality criterion implies that preferences are independent over time.

Nowadays more realistic criteria are available. In this painstaking analysis of stationary utility Koopmans [1960] presented a class of them for the case of discrete time periods; he used an assumption of limited non-complementarity over time, and showed that there exist welfare functions for which the pure rate of time preference is variable, depending on the consumption rate. Later work has shown that such utility functions exhibit the quasi-cardinal property of time-perspective [Koopmans, Diamond, and Williamson; 1964] –concept which means that as consumption programs are postponed by inserting coincident initial sections their welfare levels tend to equalize–. It was also shown that some but no al of the previous conclusions of optimal growth theory based on a constant rate of time preference hold when a member of this class of utility functions is used as a welfare criterion [Beals and Koopmans, 1969; Iwai, 1972]. In particular, optimal development programs may now depend on initial conditions.

Equivalent results for the case of continuous time have been reached by the present author [Mantel, 1967a, 1967b] but have not been widely circulated except for the analysis of a particular case in which the resulting welfare function can be explicitly represented as an integral [Mantel, 1967c]. The integrand in this version has the usual form of an instantaneous utility functions of the current consumption rate, discounted by a factor which in its exponent has the negative integral of all past discount depends on the consumption rate current at the time in which the discount rate is to be applied. It was then shown that the limiting capital-labor ratio of an optimal program in a neoclassical economy may depend on initial capital and labor endowments.

A similar explicit representation has been used by Uzawa [1968]. Nevertheless he arrives at optimal paths which in the long rung are independent of initial wealth. This is due to his particular assumption on the way in which the rate of time preference depends on the rate of consumption. He considers a positive relation, so that higher levels of consumption correspond to higher rates of time preference. This is not a realistic assumption. As Blanchard and Fischer [1991] point out, this "is not particularly attractive as a description of preferences and is not recommended for general use". Irving Fisher, the father of the creature, explains in this *Theory of Interest* [1930, page 247] that "near the minimum of subsistence... to give up one iota of this year's income in exchage for any amount promised for next year would mean too great a privation in the present, ...that is rate of time preference will gradually decrease ...that is, the larger the income, other things remaining the same, the smaller the degree of impatience".

The results for growth theory obtained in Mantel [1967c] on the contrary illustrate the use of a simplified form of such a welfare function taking into account Fisher's form for the pure rate of time preference; the qualitative behavior of optimal growth paths is there seen to be similar to that described in Beals and Koopmans [1969] and Iwai [1972] for discrete time, who also observed the multiplicity of asymptotic growth paths, with long run situations depending on the initial endowments.

Such a behavior provides an alternative explanation of the differing growth rates of different countries than that provided by Barro and Sala-i-Martin [1992]. Rather than those differences originating in technological factors such as human capital or non convexities, here they are due to preferences. Thus it can be seen that these may also lead to a "poverty trap" even in the case of a well behaved neoclassical technology.

In the present article it will be shown that the same techniques of dynamic applied in the previous study allow the analysis of a more complex preference structure, in which the welfare function cannot be written down in closed form, more in line with the two studies by Beals and Koopmans and Iwai mentioned above. Assuming time to be continuous rather than subdivided into discrete periods allows a simpler description of the optimal paths, since in that case the more powerful methods of the theory of optimal control processes are available. The advantage of the continuous time approach would stand out even more clearly in the cases in which the solutions approach the boundaries of the constraint sets –for example when there are time intervals in which consumption drops to zero– for then the behavior of the system in the discrete time case can easily become chaotic. In the continuous time case the paths approach the boundaries in essentially the same wey no matter how far they are initially from them; this is not true in the discrete time case.

Consumption programs are defined in section 1, where the structure of preference over time that will be used is presented with a list of postulates and as summary discussion of their economy significance. Section 2 describes the usual neoclassical technology and the set of feasible programs. It also defines discount factors and prices associated with feasible programs. Section 3 derives the conditions that must be satisfied by programs which stay within the restrictions imposed by the technology and resources described in section 2 and are optimal for the preferences of section 1. Section 4 presents the results of the investigation for the theory of economic growth.

In the main text only the results will be given, with some indication as to their proofs. More detailed proofs are left for the Appendix.

1. Preference over time

The present section presents briefly the structure of preference over time to be used in the sequel. A more thorough analysis has been carried out previously [Mantel, 1967a, 1970], where the gap between the two approaches –continuous vs. discrete time– has been bridged, by showing that a suitable limiting process allows one to define a utility function for continuous time with a variable rate of time preference. The main results is that the assumptions of stationarity and limited non-complementary over time imply that the prospective utility of a consumption program extending from the present to the unlimited future, can be evaluated as the initial value of the solution of a differential equation, relating the maginal increase in prospective utility due to the advancing of the program, to the level of that utility and to the instantaneous utility of the commodity bundle thereby discarded.

The property of time perspective can then be stated very simply as the condition that for some representation of the preferences tha rate of time preference be positive.

A *(time)path* or *program* is a real-valued function $z(t)$, where the non-negative arguments t represents *time*. Admissible programs are bounded and measure. A *section* of a path is its restriction to some *time duration*, the interval between two instants $0 \leq s < t$, and will be denoted by ${}_sz_t$, so that the left subscript always indicates the beginning date, the right subscript the ending date. In case $t = +\infty$ the right subscript will be omitted and one writes ${}_sz$; such a section with no ending date will be called the *tail* of the complete path ${}_0z$ starting at the *present date* $t = 0$. Of course the tail ${}_sz$ of an admissible path ${}_0z$ can by itself be transformed in to an admissible path ${}_0z'$, by dropping the initial section and advancing the tail so that $z'(t) = z(s+t)$ for all $t \geq 0$. The set of all admissible paths will be called z.

Parenthesis indicate *concatenation* of paths. For example, ${}_0z = ({}_rz'_s, {}_vz'')$ denotes a path constructed from a section of ${}_0z'$ and a tail of ${}_0z''$, that is to say, $z(t) = z'(t+r)$ for $0 < t < s-r$, and $z(t) = z''(t-(s-r)+v)$ for $t > s-r$. A repeating path ${}_0z = ({}_0z_s)$ with *period* $s > 0$ and with *pattern* ${}_0z_s$ is defined recursively by the formula ${}_0z = ({}_0z_s, {}_0z)$ and consists of the pattern repeated indefinitely, *i.e.*, ${}_0z = ({}_0z_s, {}_0z_s, \ldots)$. Similarly, for a given level z, a *constant* path ${}_0z = (z)$ has $z(t) = z$ for all instants, and cand be interpreted as a repeating path with a pattern of infinitesimally short duration.

A *consumption path* ${}_0x$ is an instance of an admissible path. The set of admissible consumption programs X consists of those admissible paths for which the consumption rate is never negative, so that $x(t) \geqslant 0$ for all t. A *welfare function –prospective utility* in Koopman's terminology– is a real valued function W defined on the set X of consumption programs. The *immediate* or *instantenaous utility* of a consumption rate c is the value of the real valued function u at c, defined to equal the prospective utility of the program offering that consumption at all times, *i.e.*,

$$u(c) = W({}_0(c)).$$

The welfare function satisfies the following postulates, which are an adaptation of those listed by Koopmans [1960] for the case in which time is subdivided into discrete periods.

P1. (Sensitivity). There exist two admissible programs ${}_0x$, ${}_0y$ which agree with each other from those time on, such that $W({}_0x) > W({}_0y)$.

This postulate means that for those two programs, there exists a moment s such that their tails agree, so that the inequality in the statement of the sensitivity postulate implies the following chain of relations,

$$W({}_0x) = W({}_0x_s, {}_sx) = W({}_0x_s, {}_sy) > W({}_0y_s, {}_sy) = W({}_0y).$$

It serves the purpose of excluding the unintersting case in which all consumption programs are equivalent to each other, which then trivially would all be optimal.

P2. (Limited non-complementary over time). For all time durations s, and for all programs $x, y \in X$, and all consumption rates b, c,

$$W({}_0(b)_s, {}_sx) \geq W({}_0(c)_s, {}_sx) \text{ implies } W({}_0(b)_s, {}_sy) \geq W({}_0(c)_s, {}_sy).$$

That is, the ordering of two initially constant programs with the same tail is not affected if their common tail is replaced by another one, as long as after the replacement both programs stll have equal ending sections.

The limited non-complementarity postulate is the central assumption which allows writing the welfare function in terms of a differential equation.

The condition that the comparison be limited to programs which are initialy constant is essential; without it, the present and the next postulates would imply that the utility function can be taken to be additive, expressed as an integral of instantaneous utilities, discounted at a constant rate, as will be shown after the statement of the next postulate.

P3. (Stationarity). For all time durations s, and all programs ${}_0x, {}_0y \in X$,

$$W(x) \geq W({}_0x_s, {}_sy) \text{ if, and only if, } W({}_sx) \geq W({}_sy).$$

Stationarity means that the ordering of two programs with a common initial segment –note that the initial segment of duration s of the first program is precisely that of the second program, that is to say ${}_0X_s$ in both cases; if the left hand side of the first inequality is written in the equivalent form $W({}_0x_s, {}_sx)$ this initial equality is perhaps more obvious– is the same as that of the two programs advanced by the time duration of the common segment, *i.e.*, is the same as the ordering of the two tails if they were to start now. The purpose of this postulate is not its realism; one might argue that future generations have different tastes, so that the evaluation of a program from their perspective is not equal to the present generation's evaluation of the same program from today's perspective if it were to start today. The reason for requiring this postulate to be satisfied is to isolate the pure time preference effect from changes in tastes, in the belief that given sufficient freedom in the choice of preferences any development path may be justified. This would then provide no proof that development paths behave differently in the long run solely on the grounds of different initial endowments in response to a variable rate of time preference.

For a sketch of the proof of the assertion that the foregoing postulates imply a constant rate of time preference, asuume temporarily that postulate 2 were strengthened as follows.

P2'. (Strong limited non-complementarity over time). For all time durations s, and for all programs $x, y \in X$, and all initial consumption segments ${}_0b_s, {}_0c_s$,

$$W({}_0b_s, {}_sx) \geq W({}_0c_s, {}_sx) \text{ implies } W({}_0b_s, {}_sy) \geq W({}_0c_s, {}_sy).$$

Then, in the first place, one should observe that this means that the ordering of the initial sections do not depend on the tails. Next, by an application of this postulate one would obtain that the inequatily.

$$W({}_0c_s, {}_sb_{2s}, {}_{2s}x) \geq W({}_0c_s, {}_sb'_{s2}, {}_{s2}x)$$

implies

$$W({}_0c_s, {}_sb_{2s}, {}_{2s}y) \geq W({}_0c_s, {}_sb'_{s2}, {}_{s2}y)$$

which by the stationarity postulate, by dropping the first section, implies

$$W({}_sb_{2s}, {}_{2s}x) \geq W({}_sb'_{s2}, {}_{s2}x)$$

which in turn, again by the stationarity postulate, by adding a different first section, implies

$$W({}_0c'_s, {}_sb_{2s}, {}_{2s}y) \geq W({}_0c'_s, {}_sb'_{s2}, {}_{s2}y)$$

showing that the ordering of the second section is independent of the first section and of the tail. Finally, the stationarity postulate by itself means that the ordering of the tails is independent of the first two sections. Having thus shown that the three sections are independent, a result by Debreu [1954] then asserts that there exists an additive representation for the welfare function. Because of stationarity all terms of the resulting sum standin a constant relation to each other, so that the discount factors correspond to a constant rate of time preference.

P4. (Extreme programs). There exist ${}_0\underline{x}, {}_0\overline{x} \in X$ such that for all ${}_0x \in X$,

$$\underline{W} = W({}_0\underline{x}) \leq W({}_0x) \leq W({}_0\overline{x}) = \overline{W}.$$

The welfare of admissible consumption programs is bounded.

It has been shown [Mantel, 1967a, 1970] that under suitable continuity assumptions these postulates imply the existence of an aggregator funcion $F(x, W)$, whose arguments are the rate of consumption x and the welfare level W which is strictly decreasing in its first argument, and –if the representation of preferences is chosen appropriately– is strictly increasing in its second argument. The aggregator function has the property that the welfare of a program can be evaluated by solving the following differential equation

$$\dot{W}(t) \equiv F(x(t), W(t)) \tag{1.1}$$

with bounded end condition

$$W \leq W(t) \leq \overline{W}$$

for its initial value $W(0)$. The solution is given by a *welfare path* ${}_0W$ such that $W(t) = W({}_tx)$ for almost all t. Note that the same symbol W is used with two different meanings: as the solution of the differential equation when its argment is the time t, and as the prospective utility or welfare when the argument ist ${}_tx$, the tail of the program starting at time t. Usually no distinction will be made since their values are equal almost everywhere.

The interpretation of differential equation (1.1) is as follows. The prospective utility of the consumption program starting at time t is $W(t)$. The program offers a consumption rate $x(t)$ at that time. The aggregator function $F(\,)$ uses this information to indicate that if those two quantities are known, advancing the program by discarding the consumption of the first instants after the current time t achieves an increase in prospective utility at the rate $\dot{W}(t)$.

For the purposes of the maximization of welfare to be carried out in thext section, it will be assumed that the following conditions hold for the aggregator function.

P5. (Utility aggregator). The *utility-aggregator* function $F: \mathbb{R}^2 \to \mathbb{R}$ is

a) continuous and twice continuosly differentiable for $x > 0$,
b) convex for all x, W, strictly so in x for all W,
c) $F_x < 0$, and $F_x(0, W) = -\infty$ for all W,
d) $1/\varepsilon \geq F_W \geq \varepsilon > 0$ for some constant ε,
e) for all $x \geq 0$ there exists W such that $F(x, W) = 0$.

It is easily verified that such an aggregator function procedures a utility function which satisfies the postulates. The level curves of a function satisfying Postulate P5 are shown in Figure 1.

In terms of the aggregator F, one can define the instantaneous utility as the solution $u(x)$ of the identify $F(x, u(x)) = 0$. This is so since a constant program is not affected by advancing it, hence $u(x) = W({}_0(x))$ so that $W({}_0(x)) = W({}_t(x))$ for all t, and thus $\dot{W}({}_t(x)) = 0$.

For uniformly bounded admissible consumption programs ${}_0x$ one has

$$W({}_0x) = \lim_{T \to \infty} W(0; T, W(T))$$

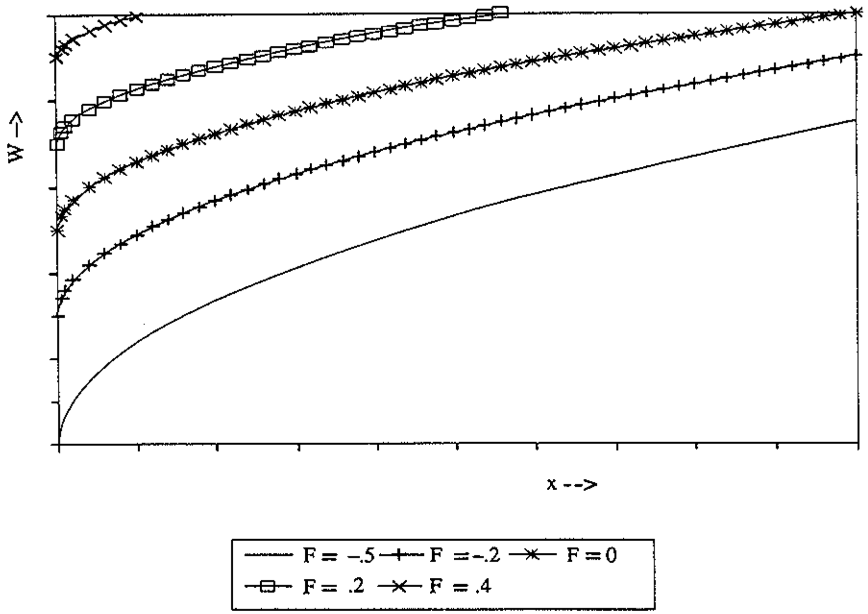
where $W(t; T, W(T))$ is a solution of the differential equation (1.1) with any end condition satisfying

$$\underline{W} \leq W(T) \leq \overline{W}.$$

We shall give $F_W(.)$ the name of *instantenous rate of time preference.* As will be seen it acts as a discount rate. Note that it is independent of the representation of preferences only for constant programs; in the general case its value depends on the utility scale. For constant programs, we define the *pure rate of time preference* as $r(x) = F_W(x, u(x))$.

FIGURE 1

AGGREGATOR FUNCTION $F(x, W)$



2. The technology and feasibility

The technology will be described by a simple neoclassical aggregate production function with the following properties.

P6. (Technology). The production function $f\colon \mathbb{R}_+ \to \mathbb{R}$ is

a) continuous, twice continuosly differentiable for $k > 0$,
b) $f(0) = 0; f'(0) > 0; f''(k) < 0$,
c) there exists a $k_m > 0$ such that $f(k_m) = 0$.

Here it is assumed that there exists only one good, used both for consumption and for accumulation. The simbol k stands for the *capital-labor ratio*, $f(.)$ for the *output-labor ratio* –the latter net of maintenance and other costs, including the investment necessary for keeping the capital-labor ratio constant–. The second assumption is standard, and states that capital is an indispensable input and that output per capita is an initially increasing and concave function of capital per capita. The last line can be justified in an economy with a growing labor force, where it is conceivable that as labor

becomes scarce it will be impossible to procedure enough to sustain the capital-labor ratio. In the sequel no reference will be made to the rate of growth of labor, which will be assumed to the constant. All relevant variables will be expressed in per capita terms.

Figure 2 shows the graph of a function satisfying Postulate P6 on the production function.

Denote the highest sustainable –"golden rule"– consumption rate by $\bar{x}$, the corresponding level of capital by $\bar{k}$, so that both quantities are positive and $f'(\bar{k}) = 0$; $\bar{x} = f(\bar{k})$.

A *capital path* is an admissible path ${}_0k$; it is *feasible for an initial capital stock k* if $k(0) = k$ and $0 \le s \le t$ implies

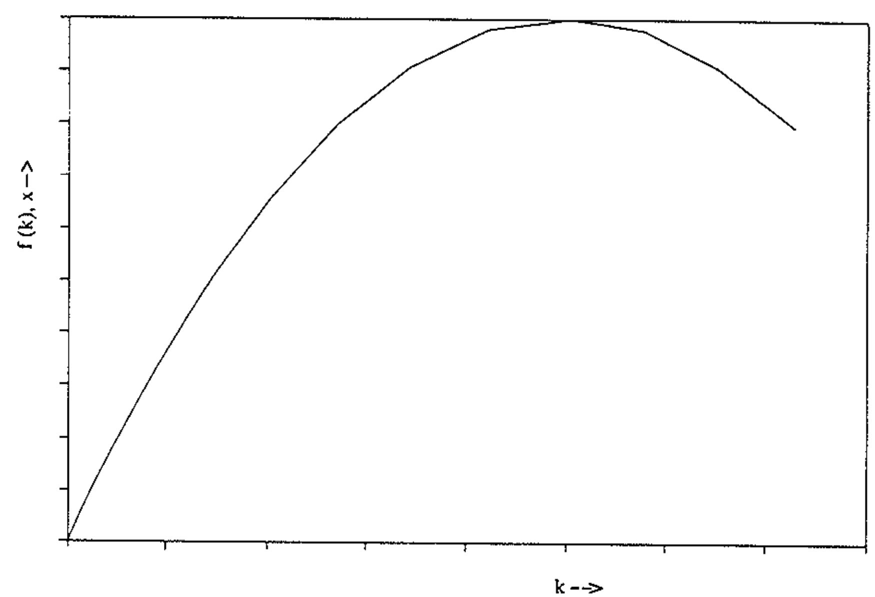
$$k(s)\, e^{-\delta(t-s)} \le k(t) \le k(s) + \int_s^t f(k(v))\, dv,$$

where $0 < \delta < \infty$ represents the rate of capital deterioration –depreciation plus the growth rate of labor–, the highest rate at which capital can be usep up. Thus a feasible capital path is differentiable almost everywhere in the sense of Lebesgue and satisfies the corresponding differential inequalities

$$-\delta k(t) \le \dot{k}(t) f(k(t)).$$

FIGURE 2

PRODUCTION FUNCTION $f(k)$



The associated consumption path ${}_0x$ satisfies

$$x(t) = f(k(t)) - \dot{k}(t), \tag{2.1}$$

so that $0 \leq x(t) \leq f(k(t)) + \delta k(t)$.

To simplify the exposition, the analysis will be restricted to those situations in which the initial capital stock is productive, *i.e.*, $0 < k(0) < k_m$. In that case feasibility implies $0 < k(t) < k_m$ for all t. Consequently the capital path –as well as the consumption path– is uniformly bounded. The problem to be solved now consist in determining the optimal feasible capital, consumption and welfare programs.

The analysis will be simplified by decomposing the maximization process into several elementary steps. With any feasible program one associates certain tentative implicit prices for the consumption good and the use of the same as capital good; these prices can then be used to compare different programs. In the end, for the optimal program, they turn out to equal the dual or co-state variables of the maximization problem.

Define the (psychological) *discount factor*, λ, and the *prices, p, q, associated* with a feasible path $({}_0W, {}_0x, {}_0k)$ as follows. The discount factor is

$$\lambda(t) = e^{-\int_0^t F_W(x(s), W(s))\, ds} \tag{2.2}$$

This expression uses the instantaneous rate of time preference F_W as a discount rate to evaluate the relative merit of events at time t as seen from the present time 0.

For the consumption good, take the discounted increase in welfare due to a marginal increase in consumption, *i.e.*

$$p(t) = \lambda(t)\,[-F_x(x(t), W(t))]. \tag{2.3}$$

For the rental price of the use of capital take the value of its marginal product at consumption prices,

$$q(t) = p(t) f'(k(t)). \tag{2.4}$$

These definitions allow the following results to be obtained.

Proposition 1. If $({}_0W, {}_0x, {}_0k)$ and $({}_0\hat{W}, {}_0\hat{x}, {}_0\hat{k})$ are feasible, then

$$W({}_0x) - W({}_0\hat{x}) \leq \int_0^\infty \hat{p}(t)\, (x(t) - \hat{x}(t))\, dt$$

This proposition –a result similar to Koopmans' [1965] proposition (F) for a constant rate of time preference– states that the difference between the welfare levels or prospective utilities of two consumption paths –the left hand side of the inequality– does not exceed the present or discounted value of the difference of the two consumption programs are evaluated at the discounted prices of the consumption good associated with the second path.

The next proposition compares the consumption programs with the corresponding capital programs.

Proposition 2. If $({}_0W, {}_0x, {}_0k)$ and $({}_0\hat{W}, {}_0\hat{x}, {}_0\hat{k})$ are feasible with the same initial capital, then

$$\int_0^\infty \hat{p}(t)(x(t) - \hat{x}(t))\, dt \leq \int_0^\infty (\hat{q}(t) + \dot{\hat{p}}(t))(k(t) - \hat{k}(t))\, dt + \lim_{t \to \infty} \hat{p}(t)\hat{k}(t).$$

This proposition –comparable to Koopmans' [1965] proposition (G) for a constant rate of time preference– states that, evaluated at the implicit prices of the second path, the present (discounted) value of the difference of the two consumption paths –the left hand side of the inequality– does not exceed the difference in the present value of the two capital services (evaluated at the price for the use of capital services $\hat{q}$) plus capital gains (due to changes in the price of the assets $\hat{p}$) –these two concepts are represente by the terms under the integral sign on the right hand side–, plus the scrap value of the final capital stock of the second path –the last term, the limity of the value of the cpaital stock as time tends to infinity–.

3. Optimality

The two propositions of the previous section lead immediately to the conditions that must be satisfied by optimal programs. Linking the two inequalities in proposition 1 and 2 together –the right hand side of the first is the left hand side of the second– the sufficiency of the condition in the next proposition should be obvious, whereas the necessity follows from the maximum principle of optimal control theory. A more intuitive argument is given below.

Proposition 3. If the rate of capital deterioration δ is sufficiently large, necessary and sufficient for the optimality of the given path $({}_0\hat{W}, {}_0\hat{x}, {}_0\hat{k})$ is that its implicit prices satisfy

$$\hat{q}(t) + \dot{\hat{p}}(t) = 0 \text{ for } t \geq 0, \tag{3.1}$$

and the transversality condition $\lim_{t \to \infty} \hat{p}(t)\dot{\hat{k}}(t) = 0$ hold.

Equation (3.1) can be rephrased as saying that the discounted price of the consumption good should fall at a rate equal to the rental price of the capital services it provides. Note that this result is in line with Koopmans' [1965] proposition (H) for a constant rate of time preference.

A heuristic argument, similar to the Keynes-Ramsey-Koopmans argument –first presented by Ramsey [1928], who attributes it to Keynes for the case of a zero rate of time preference, and later by Koopmans [1965] for a constant rate of time preference– is as follows. At any time t, increasing consumption by a fraction ε of the investment rate, $\dot{k}$, during a short time interval η means an increase in consumption of $\Delta x = \varepsilon \dot{k}$. This produces a gain in welfare equal to

$$\Delta W = \Delta (W ({}_0x) - W ({}_\eta x)) = -\eta \Delta \dot{W} = -\eta \Delta F = -\eta F_x \Delta x = -\eta F_x \varepsilon \dot{k}$$

and a loss –due to postponement of capital accumulation by a fraction ε of the time period η –equal to $\varepsilon \eta \dot{W}$. The net gain is therefore $-(F_x \dot{k} + \dot{W}) \eta \varepsilon$ and should not be positive if the capital path is optimal. Since e can have any sign, it follows that

$$\dot{W} + F_x \dot{k} = 0 \tag{3.2}$$

The foregoing argument can be shortened considerably, and perhaps made more intuitive, if one chooses the time unit to correspond to a very shor interval, say a second or a fraction thereof. One can then increase the consumption rate during the second beginning at time t by cutting investment to zero, thereby earning a welfare benefit of $-F_x \dot{k}$. The new capital stock will now be reached a second later, so that the consumption program will have to be postponed by a second at a welfare cost given by $\dot{W}$. At an optimum the net benefit is zero, so that equation (3.2) is again satisfied.

Multiplying this equation by the discount factor λ and using the definition (2.3) of the price p one then obtains $-\lambda \dot{W} + p \dot{k} = 0$, or replacing the time-derivatives from equations (1.1) and (2.1), $-\lambda F + p (f(k) - x) = 0$. Computing the derivative with respect to time t of this identity and reordering the terms gives

$$-(\lambda F_x + p) \dot{x} - (\lambda F_W + \dot{\lambda}) \dot{W} + (q + \dot{p}) \dot{k} = 0.$$

The first two terms drop out because of the definitions of p in (2.3) and λ in (2.2). Thus if the investment rate is not zero, the equality (3.1) follows.

Note that the zero net welfare benefit condition can be written as

$$p / \lambda = -F_x = \dot{W} / \dot{k} = \frac{d W}{d k}$$

which shows that the undiscounted price of the consumption good measures the welfare effect of a marginal addition to the capital stock.

Proposition 4. For any initial capital stock $0 \leq k(0) < k_m$ there exists an optimal path.

The purpose of this result is to confirm that one is not making statements non-existing items.

Proposition 5. The welfare levels of optimal programs are an increasing function of the initial capital k.

That is to say that $W({}_0x)$ increases with $k(0)$. This confirms the intuition that more resources are better.

Define a capital path to be *strictly monotone* if it is constant or either always strictly increasing or else always strictly decreasing: Then one has

Proposition 6. Optimal capital paths are strictly monotone.

In other words, under the present assumptions, optimality excludes bulges or cycles in capital programs. In this the present analysis does not differ qualitatively from the standard result obtained with a constant rate of time preference.

Proposition 7. Optimal capital paths are strictly increasing (decreasing, constant) if the marginal product of the initial capital stock exceeds (is less than, equals) the pure rate of time preference corresponding to a constant capital path equal to that initial capital stock, that is, if

$$f'(k) > (<, =) \; r[f(k)] \tag{3.3}$$

This result is also true if the rate of time preference is constant. The difference resides in that if the pure rate of time preference is constant then there is only one capital-labor ratio with a marginal product equal to it, whereas if it is decreasing there may be several solutions to the equality in relations (3.3). This central result of the present investigation is summarized in the next proposition.

Figure 3 graphs the marginal product of capital and the pure rate of time preference corresponding to stationary programs. As drawn they cross at two points giving rise to three stationary solutions, one being the origin. The arrows show that point A corresponds to an ustable situation, whereas point B and the origin are stable.

FIGURE 3

MPK AND RPT

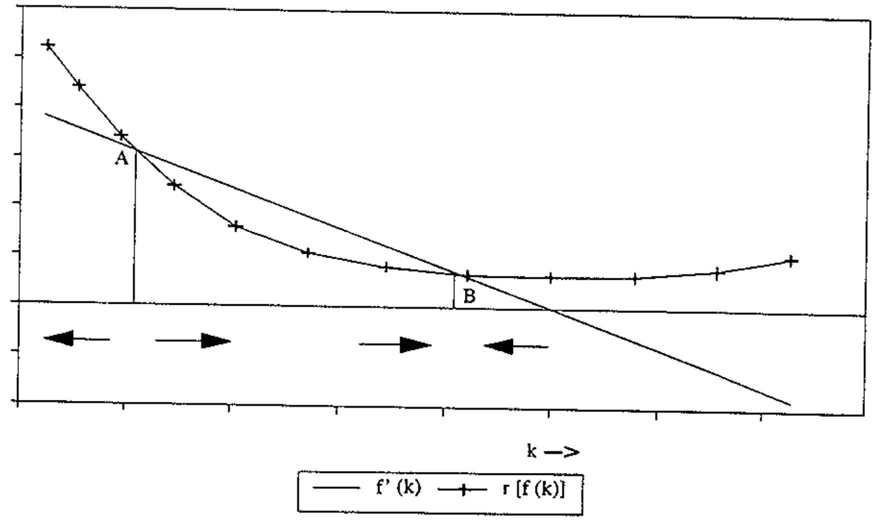
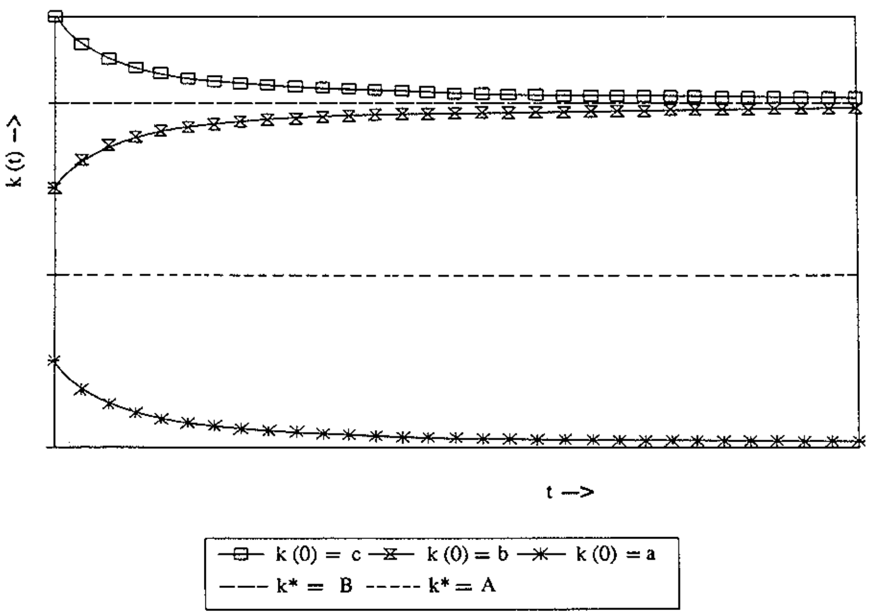


FIGURE 4

CAPITAL PROGRAMS *k* (*t*)



Proposition 8. If the initial capital stock is very large, the optimal path will be strictly decreasing. If r (0) < f' (0) and the initial capital stock is very low the path will be strictly increasing, else it will decrease toward zero. For intermediate initial capital stocks, there may be several intervals for which the path rises or for which it falls, separated by constant paths along which the pure rate of time preference equals the marginal product of capital.

Figure 4 shows the capital paths corresponding to the marginal product of capital and the rate of time preference schedules of Figure 3. The monotónicity property of Proposition 6 is illustrated, and it can be seen how the constant equilibrium paths separate those that are always increasing or always decreasing.

4. Conclusion

The present investigation started with setting out a welfare function for a planner wishing to design an optimal growth program for a neoclassical economy. "The proof of the cake is in the eating", which in the case of an economist in the position to advise the planner means that it is advisable to try out several criteria for optimal growth so as to ascertain the effects these have on the shape of the resulting optimal programs. It is

difficult to ask the planners for their preferences, so it will be simpler to deduce them from their choice among optimal paths obtained from different optimality criteria.

The steps followed in the previous sections can be summarized as follows.

In the first place, postulates for a certain structure of preferences over time have been presented, suitable for the application of the usual results of dynamic optimization when time is continuous and the planning horizon extends to the unilimited future.

The central idea is that this structure should be

a) not so simple as to reduce the welfare function to one with a constant pure rate of time preference, but
b) simple enough to be amenable to analysis, using the large body of results pertaining to optimal control theory.

In the second place, this class of welfare functions has been used to determine an optimimum in an economy described by a simple neoclassical constant returns to scale production function, and so to show the differences and similarities of the qualitative behavior of the resulting optimal trajectories of capital accumulation.

The results that have been obtained show that on the one hand there are similarities with the case of a constant rate of time preference, in that the capital paths are not of three types.

a) constant for all time, in case that initially the pure rate oftime preference coincides with the marginal productivity of capital;
b) strictly increasing, accumulatin capital by consuming less than is produced, approaching a long run capital-labor ratio asymptotically in case the pure rate of time preference falls initially short of the marginal productivity of capital;
c) strictly decreasing, decumulating capital by consuming more than is produced, again approaching a long run capital-labor ratio asymptotically, in case the pure rate of time preference exceeds initially the marginal productivity of capital.

On the other hand there are important differences:

a) In the case of a constant rate of time preference there exists a unique capital-laor ratio to which all capital programs tend in the long run independently of the initial endowment of the economy. In other words, poor societies will restrict their consumption to accumulate capital until the long run capital-labor ratio is reached, whereas rich societies will eat up their capital until that same long run capital-labor ratio is attained.
b) In the case of a variable rate of time preference –if it is falling as proposed by Irving Fisher–, as opposed to a constant rate of time preference –or increasing, as proposed by Uzawa– there may exist a multiplicity of long run relative endowments. This means that the development path of an economy depends on its initial endowments; society is not willing to disregard its past.

It seems quite reasonable to find situations in which there are at least two different capital-labor ratios at which the pure rate of time preference equals the marginal product of capital. In such a case, a very poor society may decide that the effort to accumulate capital is too high, that the benefits will take too long to be reaped, and thus embark in a high consumption program leading to a low –perhaps zero– long run capital-labor ratio. On the other hand, a somewhat richer society with an initial capital endowment exceeding some critical amount, may have sufficient incentives to decide to undertake the effort, to tighten their belts by consuming less, to accumulate and reach a long run capital-labor ratio that is higher than the present one.

More than two coincidences between the pure rate of time preference and the marginal product of capital are possible but do not seem to be plausible.

Future research on the subject analyzed in the present article might proceed along different lines.

One possibility is to relax the postulates on the preference structure. The limited non-complementarity over time does not seem to be tremendously appealing, if one considers that taking piano lessons may increase one's preferences for buying a piano. The stationarity postulate dos not take into account that future generations might have different preferences, be it due to a different appreciation of the present consumption goods, or due to their possibility of enjoying presently non-existing commodities, or, in general, to their pertaining to different environments.

Other possibilities refer to the avenues opened up by the "new" growth theory, taking into account externalities, non convexities, human capital, learning by doing or instead of doing, etc., in order to explain differences in growth rates by means of technological factors on top of preferences.

A third road would apply the ideas set forth here to equilibrium growth, assigning the preference structures here advocated to individuals instead of governments.

Other possibilities include the analysis of the effects of uncertainty on optimal growth, or an investigation of the policy prescriptions that could be derived from the considerations of models based on different preference structures.

All these developments exceed our modest intentions to provide an alternative explanation to differing growth rates than those provided by technological factors. Here it only has been pointed out that one possibility is that countries grow at different asymptotic rates because, given the technology, their initial endowments provide the incentives to do so.

When the rate of time preference is allowed to vary, a country may decide not to undertake the effort of economic development when its initial capital endowment is below some critical level, whereas if it were above that level it would be willing to sacrifice its present generation for the well-being of the future ones. It is impossible to obtain such a result with a constant rate of time preference in the case of a simple neoclassical technology.

APPENDIX

A1. Bounds on W

For any constant capital path $_0(\hat{k})$ with $\hat{k} \leq K = max\ \{k(0), k_m\}$ and any admissible path $_0k$ one has from the differential equation (1.1), because of the convexity of the aggregator function F, setting $\hat{W} = u(\hat{x})$, $\hat{x} = f(\hat{k})$, $\hat{f}' = f'(\hat{k})$, $\hat{r} = \hat{F}_W = F_W(\hat{x}, \hat{W})$, $\hat{F}_x = F_x(\hat{x}, \hat{W})$, and dropping the argument t, that

$$\dot{W} \geq \hat{F}_x (x - \hat{x}) + \hat{r}\,(W - \hat{W}).$$

Multiplying by $\hat{\lambda}$, one obtains

$$\frac{\mathrm{d}}{\mathrm{dt}} [\hat{\lambda}\,(W - \hat{W})] = \hat{\lambda}\,[\dot{W} - \hat{r}\,(W - \hat{W})] \geq \hat{\lambda}\,\hat{F}_x (x - \hat{x}) = -\hat{p}\,(x - \hat{x})$$

so that integrating between 0 and ∞, noting that the prices tend 0 as $t \to \infty$, yields

$$\begin{aligned} W(0) - \hat{W} &= [W(0) - \hat{W}] - \hat{\lambda}(\infty)\,[W(\infty) - \hat{W}] \\ &\leq \int_0^\infty \hat{p}\,(x - \hat{x})\,dt \\ &\leq \int_0^\infty \hat{p}\,(\hat{f}'\,(k - \hat{k}) - \dot{k}\,dt \\ &= \int_0^\infty (\hat{q} + \dot{\hat{p}})\,(k - \hat{k})\,dt + \hat{p}\,(0)\,(k(0) - \hat{k}) \\ &= -\hat{F}_x\,[(\hat{f}' - \hat{r}) \int_0^\infty \hat{\lambda}\,(k - \hat{k})\,dt + (k(0) - \hat{k})]. \end{aligned}$$

Since $k(t) \leq K;\ \hat{k} \geq 0;\ \int_0^\infty \hat{\lambda}\,dt = \frac{1}{\hat{r}}$, one has

$$\begin{aligned} W(0) - \hat{W} + \hat{F}_x\,[k(0) - \hat{k}] &\leq -\hat{F}_x (\hat{f}' - \hat{r}) \int_0^\infty \hat{\lambda}\,(k - \hat{k})\,dt \\ &\leq -\hat{F}_x \left| \frac{\hat{f}'}{r} - 1 \right| K. \end{aligned}$$

In particular, if $k = k(0)$ one obtains

$$W(0) - \hat{W} \leq -\hat{F}_x \left| \frac{\hat{f}'}{r} - 1 \right| K.$$

so that when $\hat{k} = k(0)$ and $\hat{f}' = \hat{r}$ the inequality $W(0) \leq \hat{W}$ follows. On the other hand, for an optimal path the inequality chanes direction, hence $W(0) = \hat{W}$. This means that if the pure rate of time preference corresponding to a consumption rate equal to the output obtained from the initial capital endowment equals the marginal productivity of that sma capital stock, then it is optimal to invest nothing forever, maintaining the initial capital endwoment intact.

A2. Maximal principle

Define $\psi(x, f, W) = F(x, W) + F_x(x, W)\ (f - x)$, i.e., the negative of the undiscounted Hamiltonian, maximized at $-F_x = p/\lambda$ with respect to x. Then $\lambda\psi = \lambda F - p\,(f - x)$, so that at an optimum –dropping the arguments of the functions– one has

$$\begin{aligned}\frac{d}{dt}(\lambda\psi) &= \frac{d}{dt}\left[\lambda F - p\,(f - x)\right]\\ &= \lambda\,(F_x\,\dot{x} + F_W\,\dot{W}) + \dot{\lambda}\,W - \dot{p}\,\dot{k} - p\,(f'\,\dot{k} - \dot{x})\\ &= (\lambda F_x + p)\,\dot{x} + (\lambda F_W + \dot{\lambda})\,\dot{W} - (\dot{p} + p f')\,\dot{k}\\ &= 0,\end{aligned}$$

because the definitions of the discount factor, prices, and proposition 3 imply that all expressions within parenthesis are zero. From the maximal principle it is known that $\lambda\,\psi = 0$ for all t.

A3. Proofs of the propositions in the text

Proof of proposition 1. Since

$$\dot{W} - \dot{\hat{W}} = F(x, W) - F(\hat{x}, \hat{W}) \geq \hat{F}_x\,(x - \hat{x}) + \hat{F}_W\,(W - \hat{W})$$

one has

$$\frac{d}{dt}\left[\hat{\lambda}\,(W - \hat{W})\right] \geq \hat{\lambda}\,\hat{F}_x\,(x - \hat{x}) = -\hat{p}\,(x - \hat{x}).$$

Integrating between 0 and ∞, since W and $\hat{W}$ are bounded in $[\underline{W}, \overline{W}]$ whereas the discount factor $\hat{\lambda} \leq e^{-\varepsilon t}$ tends to zero, the result follows.

Proof of proposition 2.

$$\begin{aligned}\hat{p}\,(x - \hat{x}) &= \hat{p}\,[f - \hat{f} - \dot{k} + \dot{\hat{k}}]\\ &\leq \hat{p}\,[\hat{f}'\,(k - \hat{k}) - (\dot{k} - \dot{\hat{k}})]\\ &= \hat{q}\,(k - \hat{k}) - \hat{p}\,(\dot{k} - \dot{\hat{k}})\end{aligned}$$

since f is concave and $p > 0$. Integrating,

$$\int_0^\infty \hat{p}\,(x - \hat{x})\,dt \leq \int_0^\infty [\hat{q}\,(k - \hat{k}) - \hat{p}\,(\dot{k} - \dot{\hat{k}})]\,dt$$

$$\leq \int_0^\infty [\hat{q}\,(k - \hat{k}) + \dot{\hat{p}}\,(k - \hat{k})]\,dt - \hat{p}\,(k - \hat{k})\Big|_0^\infty$$

$$\leq \int_0^\infty [\hat{q}\,(k - \hat{k}) + \dot{\hat{p}}\,(k - \hat{k})]\,dt + \lim_{t \to \infty} \hat{p}\,(t)\,\hat{k}\,(t)$$

The other terms drop out because $k(0) = \hat{k}(0)$, whereas $p(t) \geq 0$ and $k(t) \geq 0$ for all t.

Proof of proposition 3. Sufficiency: From proposition 1. and 2., together with (3.1), we have for any alternative feasible path with the same initial k that

$$W({}_0x) - W({}_0\hat{x}) \leq \int_0^\infty \hat{p}\,(x - \hat{x})\,dt \leq \int_0^\infty (\hat{q} + \dot{\hat{p}})\,(k - \hat{k})\,dt = 0$$

Necessity. Pontryagin's maximum principle implies the existence of two adjoint or dual functions $\lambda, p: \mathbb{R}_+ \to \mathbb{R}$ such that at any instant the Hamiltonian $H(W, k, x, \lambda, p) = -\lambda F(x, W) + p\,(f(k) - x)$ is maximal with respect to the control variable x, so that $\lambda F_x(x, W) + p = 0$. Furthermore, one also has the differential equations $\partial H/\partial k = p f'(k) = -\dot{p}$, $\partial H/\partial W = -\lambda F_W = -\dot{\lambda}$. Setting $\lambda(0) = 1$, the second of these gives the value for λ assumed before; the first one, with the definition of q, gives the required result.

As for the transversality condition, the assumptions on the utility-aggregator function imply that $r(0) \leq 1/\varepsilon$, whereas $f'(0) = +\infty$. Thus low levels of consumption imply that $r < f'$ and capital will be accumulated, so that consumption will increase setting a ceiling to the current price of the consumption good, $p/\lambda = -F_x$. Since $\lambda \leq e^{-\varepsilon t}$ and k is bounded, the result follows.

Proof of proposition 4. Our assumptions guarantee that Theorem 4, page 259, in Lee and Markus [1967] on the existence of optimal controls with magnitude constraints, is applicable.

Proof of proposition 5. Let ${}_0\hat{k}$ be an optimal capital path with initial capital $\hat{k}_0 = \hat{k}(0)$, and let $k_0 > \hat{k}_0$. Let $x(t) = \hat{x}(t) + \alpha$, for some constant $\alpha > 0$, and solve the differential equation $\dot{k} = f(k) - x$. $k(0) = k_0$. Then $\dot{k} - \dot{\hat{k}} \leq f'(k - \hat{k}) - (x - \hat{x})$ can be integrated to yeld

$$e^{-\int_0^t f'\,ds}\,(k(t) - \hat{k}(t)) - (k_0 - \hat{k}_0) \leq -\alpha \int_0^t e^{-\int_0^s f'\,dr}\,ds$$

showing that if a is sufficiently large the path for k will eventually fal below that for $\hat{k}$. Let T be the time at which both stocks are equal; then one has $W({}_0x_T, {}_T\hat{x}) > W({}_0\hat{x})$.

Proof of proposition 6. In the Appendix it is shown that if $r\,(f\,(k\,(0)\,)\,) = f'\,(k\,(0)\,)$ then the optimal capital path is constant.

Thus suppose the path is not constant. Since at the optimum $\lambda\,\dot{W} = p\,\dot{k}$, if at any time $\dot{k} = 0$ then also $\dot{W} = 0$, so that $x\,(t) = f\,(k\,(t)\,)$ and $W\,(t) - u\,(x\,(t)\,)$ and the tail of the capital path will be constant thereafter. Thus is a path starts de-(in-)creasing it cannot in-(de-)crease later on. Furthermore, if the path is constant during some interval, it must have been constant from the beginning, due to the uniqueness of the solutions to differential equations.

Proof of proposition 7. In the Appendix, section A1, the following bound was established, where ${}_0\hat{k}$ is a constant path with the same cpaital stock as the initial capital of the path ${}_0k$,

$$W\,(0) = \hat{W} \leq -\hat{F}_x\,(\hat{f}' - \hat{r}) \int_0^\infty \hat{\lambda}\,(k - \hat{k})\,dt$$

If the capital path is strictly increasing, the integral will be positive; if it is optimal, the left hand side will be positive. Thus one concludes that $\hat{f}' > \hat{r}$. Similarly, if the path is strictly decreasing the integral will be negative, so that $\hat{f}' < \hat{r}$.

Proof of proposition 8. These assertions follow from Proposition 7. The first takes into account that $\hat{r} > 0$ and $\hat{f}' \leq 0$ for $\hat{k}$ large. When $r\,(0) \leq f'\,(0) = +\infty$, by continuity, these two functions are equal at some point; in other cases they may not cross, so that capital will decumulate for ever. Multiple crossings of the corresponding graphs will produce the different intervals, with paths approaching the constant paths which are "stable" –when $\hat{r} - \hat{f}'$ increases with $\hat{k}$– and bending away from constant paths which are "unstable" –when $\hat{r} - \hat{f}'$ decreases with $\hat{k}$–.

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