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NONPARAMETRIC ESTIMATION OF MEAN AND VARIANCE AND PRICING OF SECURITIES

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Abstract

This paper develops a filtering-based framework for non-parametric estimation of parameters of a diffusion process from the conditional moments of discrete observations of the process. This method is implemented for interest rate data in the Eurodollar and long term bond markets. The resulting estimates are then used to form nonparametric univariate and bivariate interest rate models and compute prices for the short term Eurodollar interest rate futures options and long term discount bonds. The bivariate model produces prices substantially closer to the market prices.

I. Introduction

Financial economists have formulated a large number of models in continuous time. One could say that continuous time models– particularly those founded upon diffusion processes– form the bedrock of modern finance theory. Various capital asset-pricing models, option-pricing models as well as contingent-claim pricing models illustrate this. However, the estimation of such processes and the use of these estimates to price securities have assumed quite rigid functional forms for the drift and diffusion coefficients of the diffusion process. Assumptions such as constant drift and diffusion are at odds with the empirical realities of time-varying mean and variance. The interest rate market, in particular, shows significant departures from the constant mean and variance assumptions.

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III. Pricing of Securities

After estimating the conditional moments, we need to consider the pricing of securities. We consider securities whose payoff is at time T. This is a security which can be described by the following set of equations. The process followed by the underlying asset:

$$dr(t) = \mu(r(t))dt + \sigma(r(t))dB_t$$

The stochastic differential equation followed by the derivative security that is a function of the asset:

$$df(r,t)=\left[\frac{\partial f}{\partial t}+\frac{\partial f}{\partial r}\mu(r(t))+\frac{1}{2}\frac{\partial^2 f}{\partial r^2}\sigma^2(r(t))\right]dt+\sigma(r(t),t)\frac{\partial f}{\partial r}dB_t$$

Terminal condition:

$$\lim_{t\to T} f(r,t)=h(r)\ \forall r(t).$$

This pricing framework includes both discount bonds and European options. For American securities we would also need to impose the boundary conditions.

The price of this security is the diffusion process $f(r, t)$. If the volatility of the process were constant, then a riskless hedge could be formed in the manner of Black and Scholes. Then we could solve the price of the derivative security by pricing the riskless hedge. But with time-varying volatility a riskless hedge is no longer possible. Hence, to solve $f(r, t)$, we usually impose restrictions on the instantaneous return of the security, generally based on some equilibrium model.

When we impose a restriction on the instantaneous return of the security, the implication is:

Expected return of $f(r, t) = b(r, t)$

for some function $b(r, t)$

$$\Rightarrow\frac{1}{f(r,t)}\left[\frac{\partial f}{\partial t}+\frac{\partial f}{\partial r}\mu(r,t)+\frac{1}{2}\frac{\partial^2 f}{\partial r^2}\sigma^2(r,t)\right]=b(r,t)$$

$$\Rightarrow\frac{\partial f}{\partial t}+\frac{\partial f}{\partial r}\mu(r,t)+\frac{1}{2}\frac{\partial^2 f}{\partial r^2}\sigma^2(r,t)=b(r,t)f(r,t) \tag{1}$$

with the terminal condition:

$$\lim_{t\to T} f(r,t)=h(r)\ \forall r(t).$$

Following Durrett (1984) the solution to this partial differential equation is shown in Appendix B to be:

$$f(\ell,t)=E^{\ell}\left[h(r)\left\{exp\left(\int_t^T-(b(r,u)du\right)\right\}\right] \tag{2}$$

where E^{ℓ} denotes that the expectation is taken with respect to the probability density function of $r(t)$ and is taken with the starting value of the interest rate process at ℓ and with respect to time t information set.

One assumption is risk-neutrality, i.e. $b(r, t) = r(t)\ \forall t$. We make the equilibrium assumption that $b(r, t)$ can be identified in terms of the drift and diffusion of the interest rate process. This constraint implies that the return on the security equals the riskless interest rate plus the market price of interest risk times the standard deviation of the derivative security. The market price of interest risk for a security, $\lambda(r, t)$, is defined as the instantaneous excess return above the riskless rate divided by the instantaneous standard deviation over return. The market price of risk is assumed to be the same for all interest-rate derivative securities. Thus:

$$\frac{1}{f(r,t)}\left[\frac{\partial f}{\partial t}+\frac{\partial f}{\partial r}\mu(r(t))+\frac{1}{2}\frac{\partial^2 f}{\partial r^2}\sigma^2(r(t))\right]=r(t)+\lambda(r(t))\sigma(r(t)) \tag{3}$$

subject to the terminal condition that:

$$\lim_{t\to T} f(r,t)=h(r)\ \forall r(t)$$

This partial differential equation can be solved by running the diffusion process $r(t)$. This provides a probabilistic solution for the price of the security. This solution is shown in Appendix B to be:

$$f(\ell,t)=E^{\ell}\left[h(r)exp\left(\int_t^T[-r(s)-\lambda(r(s))\sigma(r(s))]ds\right)\right] \tag{4}$$

For a pure discount bond the terminal condition is:

$$f(r,T)=1\ \forall r(t) \tag{5}$$

Then the price of a discount bond is:

$$f(\ell,t)=E^{\ell}\left[exp\left(\int_t^T[-r(s)-\lambda(r(s))\sigma(r(s))]ds\right)\right] \tag{6}$$

For a European call option on the discount bound with exercise price K and expiration $Q < T$, the terminal condition is:

$$C(r, Q) = Max\ (0, f\,(r, Q) - K)\ \forall r(t) \quad (7)$$

Then the price of the option is given by:

$$C(\ell,t) = E^{\ell}\left[(Max[0, f(r,Q) - K])\ exp\left(\int_t^T [-r(s) - \lambda(r(s))\sigma(r(s))]ds\right)\right] \quad (8)$$

To derive an analytic expression for this expectation is difficult except for a few rather simple cases. An alternative is to generate diffusion processes according to the estimates of the drift and diffusions and thereby evaluated the integrals along the diffusion paths.

This is the approach taken in this paper. Therefore, the integrals are replaced by sums. The diffusion process is generated and the price of the security is computed. Then the expectation is computed as the average for a large number of diffusion process.

IV. Empirical Results

Two sets of interest rates data are used. The first is the three month Eurodollar interest rate. The second is the interest rate on 20 year constant maturity long term bonds. These are obtained from DATASTREAM. These data sets go from January 1, 1985 to November 30, 1994, a total of 2587 observations. The interest rates are converted into annualized continuous yields. The derivative security examined is the Eurodollar futures option from the Chicago Mercantile Exchange. The choice of the futures option contracts is motivated by the fact that the interest rate futures option rather than the interest rate option is the actively traded instrument for managing interest rate risk. The futures options data are on 5 exercise prices for each contract expiring in December 1994 with option premia going from April 29, 1994 to December 1, 1994. For long term discount bonds, treasury strip prices from the Wall Street Journal of December 8, 1994 are used.

Table 1 presents summary statistics for the two interest rates. Dickey-Fuller tests fail to reject the null hypothesis of unit-root for both the interest rate series. The overlapping nature of the observations induce substantial autocorrelation in the data. The tests are corrected using 20 lags. Nevertheless the the non-stationarity of the interest rate process is not proven beyond doubt. However, the increments of the discrete process do overwhelmingly reject the null of unit roots.

The drift and diffusion functions for the two interest rate series are computed from the conditional moments of the discrete observations as described in Appendix A. The estimated drift function for the two interest rates of lagged yield are shown in panels A and B of Figure 1. In both cases the conditional mean is very close to zero. However, the point estimates appear consistent with mean-reversion. When the interest rates move below their long-run mean, the drift is positive and when interest rates are above their long run, the drift is negative. The drift appears to be asymmetric with respect when the level of interest rate is above the unconditional mean versus when it is below. Panels C and D show the diffusion functions (σ^2) for

TABLE 1

SUMMARY STATISTICS

	3 Month Eurodollar Rate	Δ 3 Month Eurodollar Rate	Long Term Interest Rate	Δ Long Term Interest Rate
Mean	6.517	–0.001	8.295	–0.001
Variance	4.198	0.006	1.371	0.004
Skewness	–0.306	–0.852	0.707	–0.618
Kurtosis	–1.158	14.776	0.807	8.467
ρ_1	0.999	–0.022	0.998	–0.009
ρ_2	0.999	–0.010	0.997	–0.033
ρ_3	0.998	–0.001	0.995	–0.010
Dickey-Fuller Statistic for Non-stationarity	–1.20	–24.66	–2.89	–25.54
p-value	0.64	0.01	0.05	0.01

The "Δ 3 Month Eurodollar Rate" represents the change in the 3 Month Eurodollar Rate. The notation is similar for the "Long Term Interest Rate".
The Dickey-Fuller test is constructed using 20 lags. The p-value represents the probability of accepting the null hypothesis of nonstationarity.

the two interest rate series. The diffusion increases with the level of the interest rate, so it is consistent with a specification such as σr^{β} with $\beta > 0$. However, the rate of increase starts decreasing. The slope is much greater for the Eurodollar interest rate than the long term rate. Hence, the interest rate volatility increases much faster at the shorter end of the term structure than at the longer end.

For purposes of comparison and insight, conditional variances are plotted in Figure 2 for the level of interest rate and the increment in interest rate for both the interest rate series. Panels A and B show that the conditional variance of the level of interest rate $r(t + 1)$ conditioned on $r(t)$ and C and D show the conditional variance of $r(t + 1) - r(t)$ conditioned on $r(t)$. Both show that volatility of the interest rate increases when interest rates move away from their long run mean.

For the Eurodollar futures call options, the final payoff is given by:

$$h(r(T)) = \begin{Bmatrix} (100 - X) - r(T) & \text{if } r(T) \le 100 - X \\ 0 & \text{if } r(T) > 100 - X \end{Bmatrix}$$

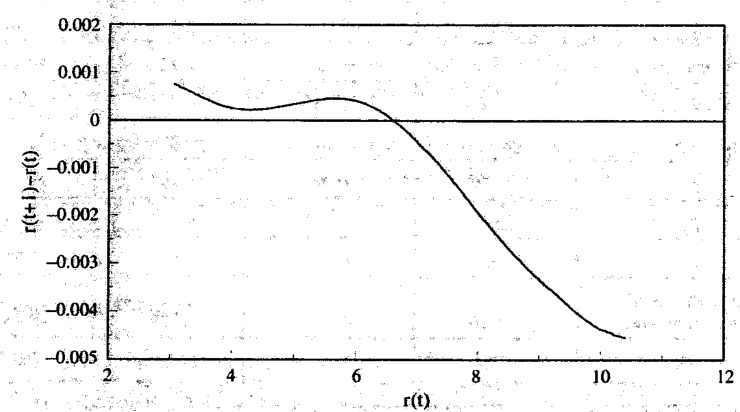
where X is the exercise price and $r(T)$ is the terminal interest rate. As a first approximation we are ignoring the early exercise option. We consider two interest rate processes. The first is a univariate process

$$dr_t = \mu(r_t)\ dt + \sigma(r_t)\ dB_t$$

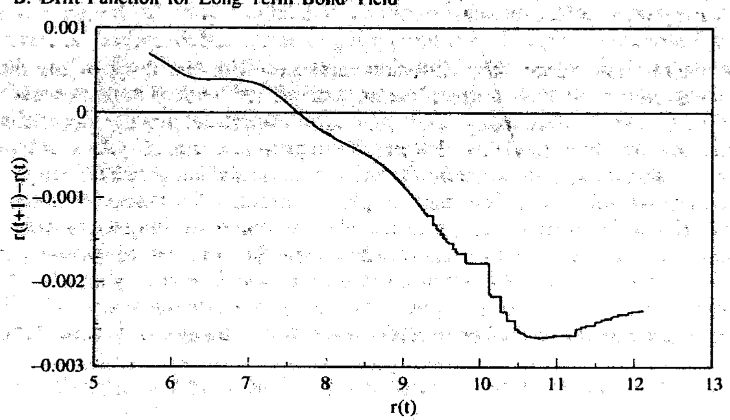
FIGURE 1

PARAMETERS OF INTEREST RATE PROCESS

A: Drift Function for 3 Month Eurodollar Yield

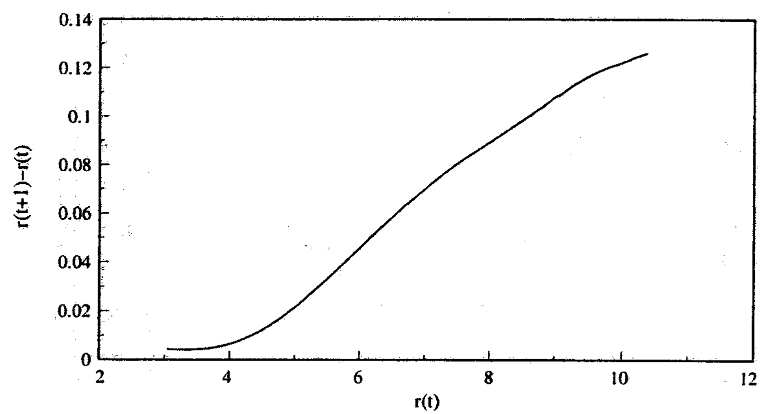


B: Drift Function for Long Term Bond Yield

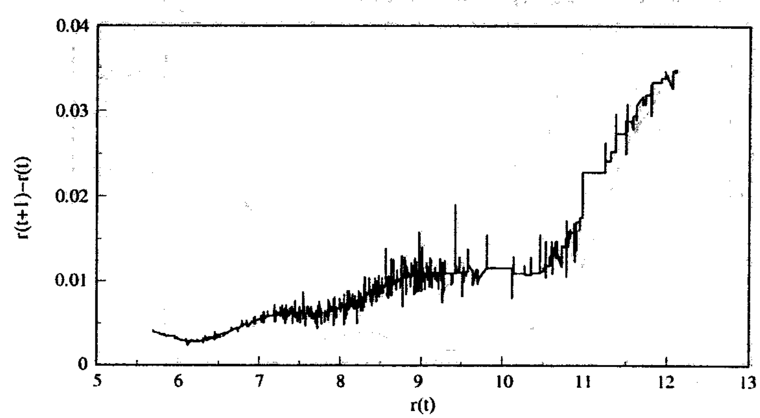


The drift functions are constructed from conditional means using adaptive kernel regression of yield changes with cross-validated bandwidth. Lagged yield is the independent variable. The yields are from January 1, 1985 to November 30, 1994, a total of 2587 observations. The weekend is treated as a single day.

C: Diffusion Function for 3 Month Eurodollar Yield



D: Diffusion Function for Long Term Bond Yield

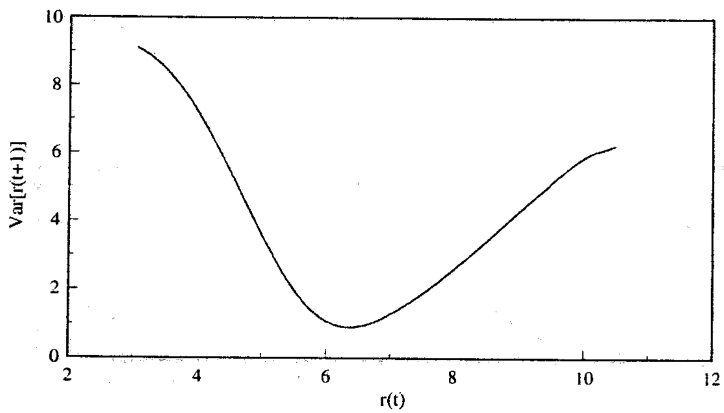


The diffusion (shown as sigma squared) functions are constructed from the conditional kernel regressions of increments of squared interest rates and covariances of drift and interest rates as shown in Appendix A. The bandwidths are cross-validated.

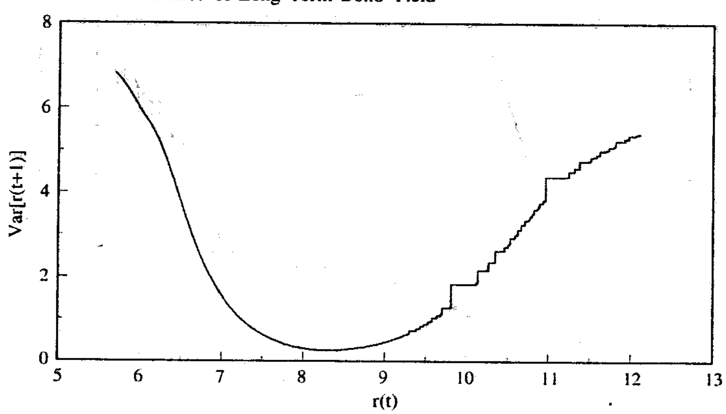
FIGURE 2

CONDITIONAL MOMENTS OF DISCRETE OBSERVATIONS

A: Conditional Variance of 3 Month Eurodollar Yield

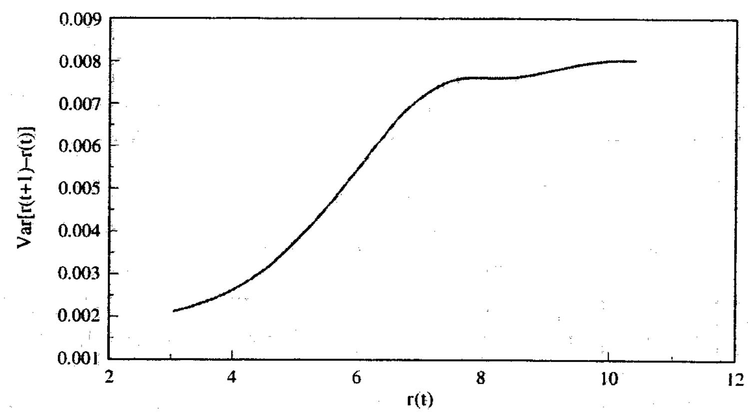


B: Conditional Variance of Long Term Bond Yield

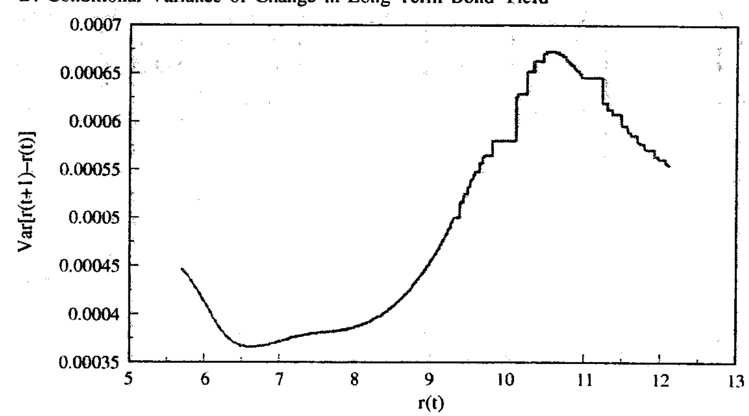


The conditional variance is constructed using adaptive kernel regression of the squared residuals from the unconditional mean of yield level with cross-validated bandwidth. The independent variable is lagged yield.

C: Conditional Variance of Change in 3 Month Eurodollar Yield



D: Conditional Variance of Change in Long Term Bond Yield



The conditional variance is constructed using adaptive kernel regression of the squared residuals from the conditional mean of change in yield with cross-validated bandwidth. The independent variable is lagged yield.

The second is a bivariate process

$$\begin{pmatrix} dr_t^s \\ dr_t^l \end{pmatrix} = \begin{bmatrix} \mu(r_t^s) & 0 \\ 0 & \mu(r_t^l) \end{bmatrix} dt + \begin{bmatrix} \sigma(r_t^s) & \sigma(r_t^s, r_t^l) \\ \sigma(r_t^s, r_t^l) & \sigma(r_t^l) \end{bmatrix} \begin{pmatrix} dB_t^s \\ dB_t^l \end{pmatrix}$$

where the superscripts s and l refer respectively to the short (Eurodollar) and long (20 year constant maturity) interest rates. I assume a interest risk premium of 0.05.[3]

Table 2 reports the computed prices of Eurodollar futures options for two maturities and three strike prices. 100 different interest rate paths are generated

TABLE 2

PERFORMANCE OF UNIVARIATE & BIVARIATE MODELS IN PRICING FUTURES OPTIONS

Strike	Days to Expiration	Short Rate	Long Rate	Market Price	Price (State=1)	Price (State=2)	Standard Futures Option Price
93.0	104	4.7500	7.7400	1.290	1.728 (1.379) (2.041)	1.727 (1.382) (2.081)	2.216
93.5	104	4.7500	7.7400	0.840	1.280 (0.938) (1.593)	1.281 (0.939) (1.629)	1.713
93.7	104	4.7500	7.7400	0.640	1.102 (0.761) (1.413)	1.102 (0.762) (1.448)	1.493
93.0	57	5.4375	7.7400	0.960	1.192 (0.863) (1.560)	1.197 (0.852) (1.571)	1.544
93.5	57	5.4375	7.7400	0.490	0.741 (0.419) (1.106)	0.747 (0.411) (1.113)	1.007
93.7	57	5.4375	7.7400	0.280	0.560 (0.240) (0.924)	0.566 (0.234) (0.930)	0.760

The table summarizes the Eurodollar futures call option prices computed using non-parametric drift and diffusion. The computation is done as an average across the interest rate paths from the drift and diffusion. The standard futures option price is computed using the assumption that Eurodollar yields are lognormally distributed. The short rate is the 3 month Eurodollar and long is the 20 year constant maturity U. S. government bond rate. The price identified as (state = 1) is computed from an interest rate model with only the short rate. The price identified as (state = 2) comes from a model with both the short and long rates. The numbers in parentheses are the 5% and 95% values of the computed price.

TABLE 3

PERFORMANCE OF UNIVARIATE AND BIVARIATE MODELS IN PRICING TREASURY STRIPS

Expiration	Maturity	Market Price	Price (State=1)	Price (State=2)	Vasicek Price
Nov 2004	10 Years	10.145	14.680	11.873	14.755
Nov 2014	20 Years	20.140	27.231	20.045	27.873
Nov 2024	30 Years	46.070	51.344	47.021	52.755

The prices of treasury strips are on December 8, 1994. The Eurodollar rate was 6.40% and the long term rate was 7.74%. The market price is computed as the midpoint between the bid and asked prices. The price from model (State = 1) is from the univariate model with nonparametric drift and diffusion. The price from model (State = 2) is from the bivariate model with nonparametric drift and diffusions. The Vasicek model prices are computed treating the 3 month rate as the short rate with the parameters computed through OLS The parameters are long-run mean = 5.459%, speed of mean reversion = 0.0009, and volatility = 0.076.

for each of the two models. For the bivariate interest rate model, the correlation between the two process is assumed to be constant. Using either the one or two interest rate models, the computed prices are greater than the market prices of the securities. However, they are closer to the market prices than the price computed using a standard futures option pricing formula assuming that the yields are lognormally distributed. The reason for the poor performance of the nonparametric model in this case can be attributed to the Federal Reserve increasing the Fed funds rate by .75% on November 15. The increase also resulted in an increase in the Eurodollar rate. The market price of the futures option appears to have incorporated the anticipated increase in the Fed funds rate.

Finally, we also examine prices for principal only treasury strips of 10, 20 and 30 years maturities. The date used is December 8, 1994. The 3 month Eurodollar and 20 year bond interest rates on that date are respectively 6.40% and 7.74%. For this prices the risk premium is assumed to be 0. For comparison, prices are also computed with the Vasicek mean-reversion interest rate model. Prices are computed for the strips using the univariate and bivariate models with nonparametric drift and diffusions. The univariate model prices are substantially closer than the market prices and are closer to the Vasicek prices. However, the bivariate model prices are quite close to the market prices. One can conjecture that a univariate model incorporates information only about the level of the interest rate whereas a bivariate model also adds information about the slope of the term-structure.

V. Conclusions and Extensions

Extensions

The present version of the paper is preliminary. Substantial improvements are possible. The construction of the confidence bands around the model based security prices will permit statistical inference. A drawback of the interest rate models estimated is that the interest rate is not being prevented from becoming negative. Future versions of the paper will consider nonparametric specifications that preclude negative interest rates.

Pricing of other derivative securities can be considered. This version of the paper has estimated the moments of the interest rate process and examined security prices implied by the estimated process. A natural complement would be to take the observed prices as given and extract the implied moments. The feasibility of such an approach will be hopefully examined in future versions, interest rate process.

Finally, the pricing approach taken has taken an equilibrium approach as given. Whether such an equilibrium approach can be avoided will be one part of future research.

Conclusions

This paper uses a filtering based approach to estimate the drift and diffusion functions for interest rates in the Eurodollar and long term US government bond markets. The estimates are then used in univariate and bivariate processes to compute the prices of Eurodollar futures options and principal only treasury strips (i.e., pure discount bonds.) For the futures options, both the interest rate models produce prices greater than the market prices. For the discount bonds, the univariate model produces model prices greater than the observed prices. However, the bivariate model produce prices quite close to the observed prices.

Notes

1 The assumption of stationarity for the increments of the process is much less restrictive than the stationarity of the process itself, as is seen for the standard Wiener process.

2 I am in the midst of computing the confidence bands.

3 This is the average of the risk premium across Eurodollar futures prices for December 1994 and March 1995 for 10 days starting May 1, 1994.

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APPENDIX A

RELATION BETWEEN THE DISCRETE AND CONTINUOUS MOMENTS

The relation between discrete and continuous moments can be established using a filtering framework. Kalman and Bucy (1961) provide the relation in case of linear processes. However, in the present case the processes are non-linear as well as non- parametric. Let $0 < t_1 < t_2 < \ldots < t_i < \ldots < T$ denote the integer partitions of $[0, T]$, the time points where observations are available. Equation 2.3 may be integrated from s to t to obtain an expression for $r(t)$ in a stochastic integral form:

$$r(t) = r(s) + \int_s^t \mu(r(u))du + \int_s^t \sigma(r(u))dB_u$$

Taking conditional expectation of this equation at time s we get.

$$E_s(r(t)) = r(s) + E_s\left(\int_s^t \mu(r(u))du\right) + E_s\left(\int_s^t \sigma(r(u))dB_u\right)$$

the conditional expectation is taken with respect to the σ-algebra $\sigma\{r(u) \mid 0 \le u \le s\}$. But since $r(t)$ is markov, the σ-algebra can be reduced to $\sigma\{r(s)\}$. By Fubini's theorem, since the integrals are bounded, we can interchange the expectations and integrals and get

$$E_s(r(t)) = r(s) + \int_s^t (E_s(\mu(r(u)))du + \int_s^t (E_s(\sigma(r(u)))dB_u$$

The second expectation on the right is of a Wiener process.
Therefore,

$$\text{(A.1)} \quad \Rightarrow E_s(r(t)) = r(s) + \int_s^t (E_s(\mu(r(u)))du$$

Now we can replace s with t_i and t with t_{i+1}, the time points where observations are available,

$$E_{t_i}(r(t_{i+l})) = r(t_i) + \int_{t_i}^{t_{i+1}} (E_{t_i}(\mu(r(u)))du$$

$$\text{(A.2)} \quad E_{t_i}[(r(t_i+1)) - r(t_i)] = \int_{t_i}^{t_{i+1}} (E_{t_i}(\mu(r(u)))du$$

Using an admittedly crude trapezoidal approximation for the integral gives us,

$$\text{(A.3)} \quad (E_{t_i}[r(t_{i+1}) - r(t_i)] \approx [E_{t_i}(\mu(r(t_{i+1})) + E_{t_i}(\mu(r(t_i))]\frac{t_{i+1} - t_i}{2} + O()$$

This is the expression for the evolution of the conditional mean.

To solve for the diffusion parameter of the interest process we follow Jazwinski (1970) (section 4.9) and obtain the evolution equation for $r^2(t)$. By Itô's rule,

$$dr^2(t) = \left[2r(t)\mu(r(t)) + \sigma^2(r(t))\right]dt + 2\sigma(r(t))dB_t$$

Integrating this equation from s to t, we get

$$r^2(t) = r^2(s) + \int_s^t \left[2r(u)\mu(r(u)) + \sigma^2(r(u))\right]du$$

$$\text{(A.4)} \quad + \int_s^t 2\sigma(r(u))]dB_u$$

Taking conditional expectation of this equation at time s we get

$$E_s[r^2(t)] = r^2(s) + \int_s^t E_s\left[2r(u)\mu(r(u)) + \sigma^2(r(u))\right]du$$

$$+ \int_s^t E_s[2\sigma(r(u))]dB_u$$

where, as before, the relevant σ-algebra is $\sigma\{r(s)\}$

$$\text{(A.5)} \quad \Rightarrow E_s[r^2(t)] = r^2(s) + \int_s^t E_s\left[2r(u)\mu(r(u)) + \sigma^2(r(u))\right]du$$

Using the trapezoidal approximation for the integrals as before,

$$E_{t_i}(r^2(t_{i+1})) - r^2(t_i) \approx [E_{t_i}(r(t_{i+1})\mu(r(t_{i+1})))] + (r(t_i)\mu(r(t_i)))$$

$$\text{(A.6)} \quad + \frac{1}{2}\left[E_{t_i}[\sigma^2(r(t_{i+1}))] + \sigma^2(r(t_i))\right]$$

Thus, we have obtained the best estimates for $\mu(r(t))$ and $\sigma(r(t))$.

APPENDIX B

DERIVATION OF THE PRICE OF A SECURITY

Consider a process $g(r, t)$ such that

(B.1) $$g(r, t) = f(r, T - t)$$

This transformation converts the terminal condition into an initial condition. The partial differential equation is transformed into the following:

(B.2) $$-\frac{\partial g}{\partial t} + \frac{\partial g}{\partial r}\mu(r,t) + \frac{1}{2}\frac{\partial^2 g}{2\partial r^2}\sigma^2(r) = b(r,t)g(r,T-t)$$

subject to the initial condition that:

$$\lim_{t \to 0} g(r,t) = h(r) \quad \forall r(t)$$

This partial differential equation can also be written as:

(B.3) $$-\frac{\partial g}{\partial t} + \mathcal{L}g = b(r,t)g(r,T-t)$$
$$\Rightarrow \frac{\partial g}{\partial t} = \mathcal{L}g - b(r,t)g(r,T-t)$$

where $\mathcal{L}$ is the elliptic operator. This partial differential equation can be solved by running the Itô process $r(t)$ which has the drift $\mu(r, t)$ and the diffusion $\sigma(r, t)$. Durrett (1984) provides the solution for both the homogeneous and inhomogeneous cases of this equation The solution to the inhomogeneous case is not difficult. This consists of evaluating the inhomogeneous part along the path of the Itô process. Thus, in case of the general partial differential equation:

$$\frac{\partial g}{\partial t} = \mathcal{L}g - b(r,t)g(r,T-t)$$

the solution is given by:

(B.4) $$g(\ell,t) = f(\ell,T-t) = E^{\ell}\left[h(r)\, exp\left(\int_T^t -(b(r,s)ds\right)\right]$$

where E^{ℓ} denotes that the expectation is taken with respect to the probability density generated by r(t) and is taken with the starting value of the interest rate process at ℓ.

For an equilibrium model based restriction on the returns we get:

(B.5) $$\frac{\partial f}{\partial t} + \frac{\partial f}{\partial r}\mu(r(t)) + \frac{1}{2}\frac{\partial^2 f}{2\partial r^2}\sigma^2(r) = r(t)f(r,t) + \lambda(r(t))\sigma(r(t))\frac{\partial f}{\partial r}$$

subject to the terminal condition that:

$$\lim_{t \to T} f(r,t) = h(r) \quad \forall r(t)$$

Transforming the partial differential equation as before, using

$$g(r,t) = f(r,t-T)$$

we get:

(B.6) $$-\frac{\partial g}{\partial t} + \frac{\partial g}{\partial r}\mu(r,t) + \frac{1}{2}\frac{\partial^2 g}{\partial r^2}\sigma^2(r) = r(t)g(r,T-t) + \lambda(r,t)\sigma(r,t)\frac{\partial g}{\partial r}$$

subject to the initial condition that:

$$\lim_{t \to 0} g(r,t) = h(r) \quad \forall r(t)$$

As before, we write the partial differential equation using the elliptic operator $\mathcal{L}$ as:

(B.7) $$\frac{\partial g}{\partial t} = \mathcal{L}g - r(t)g(r,T-t) - \lambda(r,t)\sigma(r,t)\frac{\partial g}{\partial r}$$

There are two possible ways to solve this equation. One would be to use a risk adjusted process $r^*(t)$ with the same diffusion but a different drift. This would transform the partial differential equation into

(B.8) $$\frac{\partial g}{\partial t} = \mathcal{L}^* g - r(t)g(r,T-t)$$

where, $\mathcal{L}^* g = (\mu(r,t) + \lambda(r,t)\sigma(r,t))\frac{\partial g}{\partial r} + \frac{1}{2}\sigma^2(r,t)\frac{\partial^2 g}{\partial r^2}$

Thus, we have a transformed process $r^*(t)$ with the drift $\mu(r, t) + \lambda(r, t)\sigma(r, t)$ and diffusion $\sigma(r, t)$. Then, as in the risk neutral case, the solution of the partial differential equation is given by:

(B.9) $$f(\ell,t) = g(\ell,T-t) = E_*^{\ell}\left[h(r)E_t\left\{exp\left(\int_t^T -r^*(s)ds\right)\right\}\right]$$

where again E_*^{ℓ} denotes that the expectation is taken with respect to the probability measure generated by $r^*(t)$. By generating different interest rate paths for $r^*(t)$ with the starting value of $r^*(t)$ at ℓ we can numerically evaluate the expectation.