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ESTIMATION OF A STOCHASTIC-VOLATILITY JUMP-DIFFUSION MODEL

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Abstract

This paper makes two contributions: (1) it presents estimates of a continuous-time stochastic-volatility jump-diffusion process (SVJD) using a simulation-based estimator, and (2) it shows that misspecified models that allow for jumps, but not stochastic volatility, can give very bad estimates of the true process.
Simulation-based estimation is a very flexible and powerful technique. It is ideally suited to high frequency financial data. It can estimate models with intractable likelihood functions, and since the simulations can be performed in (essentially) continuous-time the estimates are consistent estimates of the parameters of the continuous-time process.

I. Introduction

Financial economists achieved unprecedented success over the last twenty-five years using simple diffusion models to approximate the stochastic process for returns on financial assets. The so-called "volatility smiles and smirks" computed using the volatility implied by the venerable Black-Scholes model reveal, how-

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ever, that a simple geometric Brownian motion process misses some important features of the data. High frequency returns data display excess kurtosis (fat tailed distributions), skewness, and volatility clustering. Capturing these essential characteristics with a tractable parsimonious parametric model is difficult.

Until recently, applied financial economists were forced to selection from an unappetizing menu. They could choose a process that made it relatively easy to price options, or they could choose a process that was relatively easy to estimate. Option pricing theorists added jumps and stochastic volatility to the standard geometric Brownian motion process. The additional stochastic differential equation –a second state variable– in volatility gives a tractable system of differential equations for pricing options. But, estimating stochastic volatility models is extremely challenging. Stochastic volatility is a latent variable with no closed-form representation for the likelihood function. Econometricians approximated volatility clustering by representing returns as a generalized autoregressive-conditional-heterskedastic (GARCH) process. GARCH processes have a tractable likelihood function for estimation. But in general, GARCH processes do not have a diffusion process as their continuous-time limit.

Recent advances in computing and econometrics offer a better selection. This paper presents estimates of the Norwegian Kroner–British pound exchange rate as a stochastic-volatility jump-diffusion process (SVJD) using a simulation-based estimator. Simulation-based estimation is extremely general and flexible, but computationally intensive. We use a simple specification with constant jump intensity and a mean-reverting process for volatility. This is the stylized specification from option pricing models. Although the specification is simple, the model is not rejected by the data. A specification that allows for volatility clustering and jumps is crucial because geometric Browning motion cannot match the higher sample moments in the data. Just adding jumps does not give an adequate approximation. The jump diffusion specification tries to capture the volatility clustering by splitting the data into a high and low volatility regime and ignores the jumps. Monte Carlo evidence confirms that the pure jump model is badly biased when there are jumps and stochastic volatility.

Estimation of SVJD processes is new. Duffie, Pan, and Singleton (1998) and Chernov, Gallant, Ghysels, and Tauchen (1999) use a simulation-based estimator to estimate a complex model with conditional jump intensity and stochastic volatility. Bates (2000) estimates the SVJD process for S&P futures prices implicit in futures option prices.

The outline of the paper is as follows. Section II shows the data and sample statistics. Section III presents estimates of the daily "returns" process (log differences) of the Norwegian-British exchange rate for three nested diffusion processes in popular financial economics: geometric Brownian motion, geometric Brownian motion plus Poisson distributed jumps (jump-diffusion), and a jump-diffusion process with stochastic volatility.

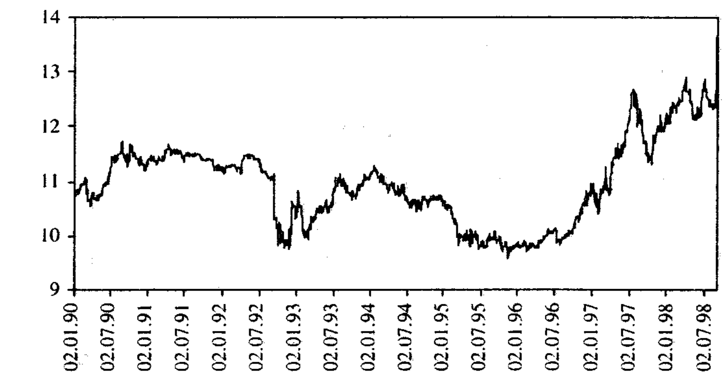
Section IV presents Monte Carlo evidence. We generate data from a stochastic-volatility jump-diffusion process and estimate a SVJD model with the simulation-based estimator and a misspecified jump-diffusion model by maximum likelihood. The small sample distribution of the estimates for the correctly specified model looks pretty good for the stochastic volatility equation. But the Monte Carlo evidence also shows that a sample size of 2000 –daily observations for a decade– is not large enough to pin down the parameters of the jump distribution accurately. The misspecified jump-diffusion model badly overestimates the jump probability and underestimates volatility of the jump and the unconditional variance of the process. It ignores the jump, and fits the stochastic volatility as a high and low volatility regime. Section V has the conclusions.

II. Data

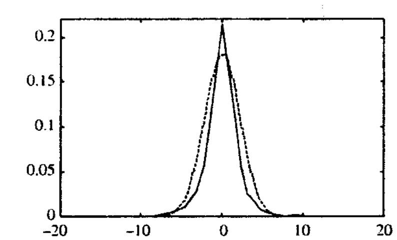
Figure 2.1 shows the time-series plot and empirical density relative to a normal density for daily returns (difference in logarithm of the daily exchange rates) on the Norwegian Kroner-British Pound exchange rate[1] from January 1990 through August 1998.

FIGURE 2.1

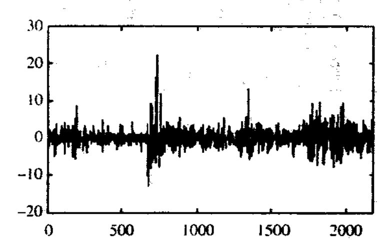
TIME-SERIES OF NOK/GBP EXCHANGE RATE



EMPIRICAL DENSITY FUNCTION FOR THE NOK/GBP RETURNS VS. THE NORMAL DISTRIBUTION



LOGARITHMIC RETURN FROM 1990 TO 1998 NOK/GBP



The Norwegian exchange rate data display the generic characteristics of foreign exchange data–excess kurtosis, skewness, potential jumps, and volatility clustering. Many studies document these stylized facts in other foreign currency markets, e.g. see, Anderson, et al. (1999), Bates (1996), and Jorion (1988).

Table 2.1 gives estimates of the first four unconditional sample moments[2] and the standard deviations of the estimates.

TABLE 2.1

ESTIMATES OF THE UNCONDITIONAL MOMENTS

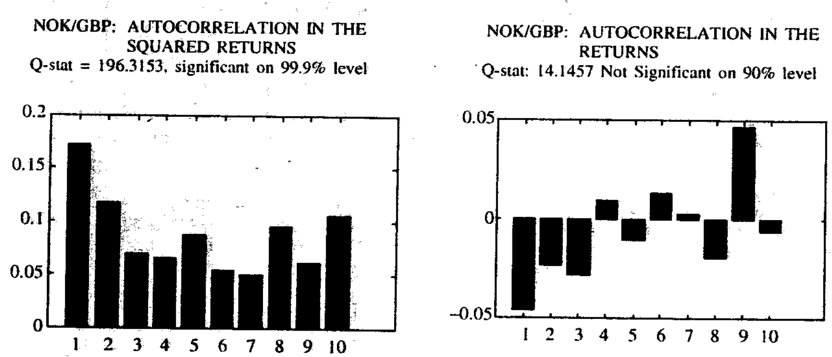
	Mean	Variance	Skewness	Excess kurtosis
Estimate	0.0402	4.8633	0.8178	10.8507
Standard error of estimate	0.0472	0.1041	0.0175	0.2967

The distribution is slightly positively skewed, 0.82, and has substantial excess kurtosis (fatter tails than a normal) of 10.85.

Figure 2.2 shows the estimated autocorrelation in returns and the autocorrelation in the squared returns.

FIGURE 2.2

AUTOCORRELATIONS



There is no significant autocorrelation in returns. There is significant positive autocorrelation in the squared returns –high volatility is followed by high volatility and vice versa.

III. Estimation

This section presents the estimates from three increasing more complicated popular specifications in financial economics: geometric Brownian motion (GBM), geometric Brownian motion plus a jump process (JD), and stochastic-volatility plus a jump diffusion process (SVJD). The goal is to find an adequate approximation to the data with the most parsimonious representation.

Returns in a GBM process are normally distributed. A normal distribution is symmetric (not skewed) and has no excess kurtosis (excess kurtosis is measured relative to a normal). By construction the GBM process matches the first two unconditional moments and cannot match the higher moments in the data. A jump diffusion process is leptokurtic and can be skewed. A jump diffusion process could match the unconditional moments. Our estimates of the jump diffusion process match the first two sample moments and generate about ¾ of the required excess kurtosis. The estimates, however, are not reasonable. The jump probability (probability of an unusual event) is 33% each day. The jump diffusion estimates try to pick up the volatility clustering in the data by dividing the model into a "high volatility regime"–the jump regime– and a low volatility regime. The stochastic volatility jump diffusion model could match the conditional and unconditional moments. Our estimates match the first two unconditional moments and generate 90% of the sample excess kurtosis. A formal specification test does not reject the SVJD model.

3.1 Geometric Brownian motion

Geometric Brownian motion[3] is the simplest and probably most popular specification in financial models. The venerable Black-Scholes option-pricing model assumes the underlying state variable follows GBM. GBM specifies that the instantaneous percentage change in the exchange rate has a constant drift, μ_B, and volatility, σ_B,

$$\frac{dS}{S} = \mu_B dt + \sigma_B dW \qquad 3.1.1$$

The error, dW, is a standard Weiner process.

Under Geometric Brownian motion the time-t exchange rate evolves according to,

$$S_t = S_0 \; exp\{\mu_B t + \sigma_B W_t\}.$$

Let $\tau > 0$ denote the interval between observations. Then, the τ period logarithmic return,

$$\ln\left({}^{S_{t+\tau}}\!/\!_{S_t}\right) \equiv x(\tau) = \mu_B \tau + \sigma_B (W_{t+\tau} - W_t),$$

is normally distributed,

$$x(\tau) \sim N(\mu_B\tau, \sigma_B^2\tau) \qquad 3.1.2$$

with a mean and variance proportional to the observation interval. This follows since the difference $(W_{t+\tau}-W_t)$ in the Weiner process is normally distributed with mean zero and variance τ.

Defining the unit interval as a day,

$$\begin{aligned} \ln S_{t+1} - \ln S_t &= \mu_B + \sigma_B W_1 \\ W_1 &\sim IIN(0,1) \end{aligned} \qquad 3.1.3$$

gives a discrete-time version of GBM that matches the data frequency.

Table 3.1.1 shows the parameter estimates.

TABLE 3.1.1

PARAMETER ESTIMATES GBM

	μ_B	σ_B
Parameters	0.0402	2.2047
Standard Errors	0.0472	0.0334

Matching the moments

The OLS estimates of the drift and volatility (square root of the variance) are unconstrained estimates of the first two unconditional sample moments, so they match the estimates in Table 2.1 exactly. The GBM model specifies that returns are iid normally distributed, equation 3.1.3. By definition the GBM specification cannot match the higher unconditional, or conditional, moments of the empirical distribution of the exchange rate.

3.2 Geometric Brownian motion plus jump

Merton (1976) added Poisson jumps to a standard GBM process to approximate the movement of stock prices subject to occasional discontinuous breaks,

$$\frac{dS}{S} = \mu_B dt + \sigma_B dW + kdq \qquad 3.2.1$$

Here dq is a Poisson counter with intensity λ, i.e., Prob $(dq = 1) = \lambda dt$, and k is a draw from normal distribution,

$$k \sim N(\mu_J, \sigma_J^2)$$

The jump-diffusion process is leptokurtic (positive excess kurtosis) and can be skewed.

The logarithmic return for any day contains two components,

$$\ln S_{t+1} - \ln S_t \equiv y_{t+1} = \begin{cases} x & \text{if Q = 0} \\ x + k_1 + k_2 + ..k_Q & \text{if Q} \geq 1 \end{cases} \qquad 3.2.2$$

a draw from the GBM process, x, plus possible draws $k_1, k_2 ..$, from the jump process. A draw from the Poisson process determines the number of draws from the jump process, k, each day,

$$\text{Prob}\ (Q = q) = \left(\frac{e^{-\lambda}\lambda^q}{q!} \right) \qquad 3.2.3$$

We estimated the parameters of the jump-diffusion process by maximum likelihood. The log-likelihood function is,

$$l(\theta, y) = \sum_{t=0}^{T} \ln \left[\sum_{q=0}^{Q} \frac{e^{-\lambda}\lambda^q}{q!} \frac{1}{\sqrt{2\pi(\sigma_B^2 + q\sigma_J^2)}} \exp\left(\frac{-(y_{t+1} - \mu_B - q\mu_J)^2}{2(\sigma_B^2 + q\sigma_J^2)} \right) \right],$$

the sum of the logs of sums of exponentials weighted by the Poisson probabilities. The scores are messy nonlinear functions of the unknown parameters, θ.[4] The maximum likelihood estimates must be computed numerically.

We computed the estimates using the optimization toolbox in MATLAB. The likelihood function is not well-behaved –as one would expect if there are occasional discontinuous jumps. There are local maxima and the algorithm did not always converge. A run took approximately one hour on a Pentium II 266 MHz PC. Table 3.2 gives the parameter estimates and sample statistics.

TABLE 3.2

PARAMETER ESTIMATES JUMP-DIFFUSION

	μ_B	σ_B	μ_J	σ_J	λ
Parameter	0.0702	1.2747	−0.0897	2.9815	0.3354
Standard error	0.0421	0.0527	0.1459	0.1487	0.0455

The jump diffusion model nests the GBM specification. It fits the data better than the simpler model. If we use a likelihood ratio test as a diagnostic, it rejects the null of no jumps –ie, the GBM specification– at the 1% level. Jumps help explain some of the features of the data.

Matching the moments

The unconditional mean of the jump-diffusion process equals,

$$\mu = \mu_B + \lambda\mu_J = 0.04$$

the mean of the GBM process plus the mean of the jump process times the probability of a jump. The drift of the GBM portion is greater than the unconditional mean (.07 > .04), but the mean of the jump process is negative and the estimate of the jump frequency, λ, (0.33) is (unbelievably) large and significant. The sum matches the estimate of the unconditional first moment almost exactly.

Das and Sundaram, Section III A, give the formulas to calculate the higher moments.

The variance of the jump diffusion process has two components: the normal times volatility of the GBM component, plus the jump component,

$$\sigma^2 = \sigma_B{}^2 + \lambda(\mu_J{}^2 + \sigma_J{}^2) = 4.61$$

The unconditional variance implied by jump diffusion estimates is only slightly below the two standard deviation confidence interval for the sample variance (4.66) in Table 2.1. The estimate of the contribution to volatility of the GBM portion of the process, σ_B, drops dramatically from *0.22%* to *0.13%* but remains very significant. The estimate of the volatility of the jump distribution is fairly large, *0.3%*, and significant. The probability of a discontinuous jump each day is 1/3.

The jump-diffusion model could, but does not generate skewness,

$$\text{Skewness}(y) = \left[\frac{\lambda(\mu_J{}^3 + 3\mu_J\sigma_J{}^2)}{(\sigma_B{}^2 + \lambda\sigma_J{}^2 + \lambda\mu_J{}^2)^{3/2}}\right] = -0.0811.$$

The unconstrained estimate of the skewness in the sample is positive, 0.81, and significant.

The jump-diffusion model generates 60% of the estimated sample excess kurtosis,

$$\text{Kurtosis}(y) = 3 + \left[\frac{\lambda(\mu_J{}^4 + 6\mu_J{}^2\sigma_J{}^2 + 3\sigma_J{}^4)}{(\sigma_B{}^2 + \lambda\sigma_J{}^2 + \lambda\mu_J{}^2)^2}\right] = 6.75.$$

which is far below the two standard deviation confidence band of 10.85 ± 0.6.

Discussion

The estimated jump-diffusion process is a poor approximation to the data. The jump-diffusion process in equations 3.2.2 and 3.2.3 is independently and identically distributed. To capture the persistence in volatility the estimator tries to split the data into high and low volatility regimes. It labels the low volatility regime the GBM process and the high volatility regime the jump. The high estimated jump probability (1/3 each day) is roughly the fraction of the sample with high volatility.

3.3 Stochastic volatility jump diffusion

3.3.1 The model

Volatility clustering is an important feature of the data. Stochastic volatility is a natural extension of the diffusion models widely applied in the asset pricing literature. Hull and White (1987), Melino and Turnbull (1990), Wiggins (1987), and others generalized the traditional GBM specification by making volatility stochastic. We add stochastic volatility to the jump-diffusion model,

$$\frac{dS}{S} = \mu_B dt + h dW + k dq$$

$$d\ln h^2 = b(\mu_h - \ln h^2)dt + c dZ \qquad 3.3.1$$

$$= a dt - b\ln h^2 dt + c dZ$$

$$k \sim N(\mu_J, \sigma_J{}^2)$$

Here the logarithm of the variance, h^2, follows a mean-reverting process with an independent Weiner error, *dZ*.

Estimation technique

Estimating the SVJD model presents two challenges. (1) The model is in continuous-time, and no closed-form expression exists for the discrete representation. And, (2) stochastic volatility is a latent variable, and no closed-form expression exists for the likelihood function.

3.3.2 Simulation-based estimation

We estimate a stochastic-volatility jump-diffusion process using a simulation-based technique introduced by McFadden (1989) and Pakes and Pollard (1989).

The idea behind simulation based estimation is disarmingly simple and extremely powerful. Mother Nature draws a sample that we observe, e.g., the NOK/GBP sample in Figure 2.1, from an unobservable data generation process. Maximum likelihood techniques assume a probability density function for the data generation process and choose parameters of the density that maximize the likelihood of the observed sample. Simulation-based methods assume the model is the data generation process. Equations 3.3.1 implicitly define the joint and conditional densities for returns and volatility. For a set of parameters, $\theta^i = \{\mu^i, \lambda^i, \mu_J^i, \sigma_J^i, a^i, b^i, c^i\}$, one can generate a simulated sample, $y^s(\theta^i)$. Choosing the parameter vector, $\hat{\theta}$, so that the simulated sample "matches" the observed sample, gives the parameter estimates of the data generation process.

Efficient method of moments

Defining what it means to "match" the sample data defines the estimator. We use the estimator called "efficient method of moments" by Gallant and Tauchen (1996) or "indirect inference" by Gourieroux and Monfort (1996) to choose the parameters.[5] The null hypothesis is that the observed sample data, $y_t(\theta)$, are drawn from the data generation process in equations 3.1.1 parameterized by the unknown parameter vector, θ.

The indirect inference methodology chooses an "auxiliary" model, or "score generator." The auxiliary model is a descriptive statistic, e.g., an autoregression, that must capture the key features of the data. Let,

$$\sum_{t=1}^{T} l(y_t(\theta);\beta) = \sum_{t}^{T} \log f^a(y_t(\theta);\beta)$$

denote the pseudo-log likelihood function for the auxiliary model. β are the parameters of the auxiliary model. The scores of the auxiliary model,

$$\sum_{t=1}^{T} l_\beta(y_t(\theta);\beta) = \sum_{t}^{T} \frac{\partial \log f^a(y_t(\theta);\beta)}{\partial \beta} = 0$$

evaluated at the pseudo-maximum likelihood estimates, $\hat{\beta}_T$, using the observed sample data equal zero by maximization.

Gallant and Tauchen's technique is to choose a parameter vector, $\hat{\theta}_{NT}$, that makes the scores of the auxiliary model *evaluated with the simulated data at the pseudo-maximum likelihood estimate*, $\hat{\beta}_T$, as close as possible to zero. That is, choose a set of parameters so that the simulated samples match the observed data. They show that, under fairly general conditions, the estimates are consistent and, under more restrictive conditions, efficient. (See Gourieroux and Monfort, Chapter 4).

Specifically, simulate the model, equation 3.1.1, for a given parameter vector, θ^i, to generate a sample, y_t^s, t = 1, ...,T. Then, evaluate the scores using the simulated data and the maximum likelihood estimates from the auxiliary model,

$$\sum_{t=1}^{T} l_\beta(y^s{}_t(\theta);\hat{\beta}_T)$$

If the parameters, θ^i, were the parameters of the data generating process and there were no sampling error, then the scores would equal zero. Increasing the sample size reduces the sampling error. So increase the sample size by drawing lots of samples, N,

$$\sum_{s=1}^{N} \sum_{t=1}^{T} l_\beta(y^s{}_t(\theta);\hat{\beta}_T).$$

Then find the best parameter vector, θ.

The efficient method of moments estimator chooses the parameter vector, θ, that makes the weighted average of the scores as close to zero as possible,

$$\hat{\theta}_{NT}(I_0) = \arg\min_\theta \left[\sum_{s=1}^{N}\sum_{t=1}^{T} l_\beta(y^s{}_t(\theta);\hat{\beta}_T)\right]' I_0^{-1} \left[\sum_{s=1}^{N}\sum_{t=1}^{T} l_\beta(y^s{}_t(\theta);\hat{\beta}_T)\right] \qquad 3.3.2$$

where I_0 is the information matrix for the auxiliary model.

The information matrix I_0 can be consistently estimated with a two-pass GMM-type procedure. In the first pass use an arbitrary positive definite weighting matrix, $\tilde{I}_0$, in 3.3.2 and get consistent, but inefficient parameter estimates, $\tilde{\theta}$. Next, use the consistent estimates to estimate the information matrix,

$$\hat{I}_0 = \frac{1}{NT}\sum_{s=1}^{N}\sum_{t=1}^{T}\left[l_\beta(y^s{}_t(\tilde{\theta});\hat{\beta}_T), l_\beta(y^s{}_t(\tilde{\theta});\hat{\beta}_T)'\right]$$

In the second pass use the estimate of "optimal" weighting matrix, $\hat{I}_0$, in the loss function 3.3.2.

Asymptotic distribution

Gallant and Tauchen (1996) and Gourieroux & Monfort, Section 4.2.3, give the asymptotic distribution of the estimates,

$$\begin{aligned} &\sqrt{T}(\hat{\theta}_{NT}(I_0) - \theta_0) \to N(0, W(N, I_0)) \\ &W(N, I_0) = \left(1 + \tfrac{1}{N}\right) D' I_0^{-1} D \end{aligned} \qquad 3.3.3$$

where,

$$D \equiv \frac{\partial^2 l}{\partial \beta \partial \theta}[\theta_0, \beta_0],$$

is the partial derivatives of the scores with respect to the parameter vector θ. D can be computed numerically.

3.3.3 Simulated sample estimates of the SVJD process

A critical step in the application is choosing a suitable auxiliary model. The auxiliary model must capture the essential characteristics of the data or it will not identify[6] the parameters of the data generation process. And it must be easy to evaluate the scores, or the problem becomes computationally infeasible.

Auxiliary model

In our application we only observe the exchange rate return process. We choose a mixture of normals to represent the jump-diffusion process and an autoregression of the squared residuals from this process to capture the volatility clustering. The auxiliary model has no latent variable.

Mixture of normals

The jump diffusion process is a geometric Brownian motion process with discrete jumps occurring at Poisson-distributed time-intervals. The jumps are normally distributed. We simplify the process by only allowing a single jump each day. The simplified process can be written as a mixture of two normals with a binomial mixing distribution. The log-likelihood function is,

$$\sum_{t=1}^{T} \log\left\{ p\frac{1}{\sqrt{2\pi}\sigma_1} exp\left[\frac{-(y_t-\mu_1)^2}{2\sigma_1^{\,2}}\right] + (1-p)\frac{1}{\sqrt{2\pi}\sigma_2} exp\left[\frac{-(y_t-\mu_2)^2}{2\sigma_2^{\,2}}\right]\right\}$$

where p is the binomial mixing probability and μ_i, σ_i^2, i = 1,2 are the parameters of the normal distributions. The mixture of normals is distributed iid. It does not capture the volatility clustering.

Autoregression

To capture the volatility clustering we run an autoregression on the squared residuals from the mixture process. Let u denote the residual from mixture of normals model. The autoregression,

$$\ln u_t^{\,2} = b_0 + \sum_{i=1}^{10} b_i \ln u_{t-i}^2 + e_t$$

picks up some of the volatility clustering.[7]

Simulation model

The discrete version of the stochastic volatility jump diffusion process is,

$$\ln\left(\frac{S_{j+1}}{S_j}\right) = \frac{\mu_d}{\delta} + h_{j+1}w_{j+1}\sqrt{1/\delta} + kq_{j+1}$$
$$\ln\left(\frac{h_{j+1}^2}{h_j^2}\right) = \frac{a}{\delta} - \frac{b}{\delta}\ln h_j^2 + cz_{j+1}\sqrt{1/\delta} \qquad 3.3.5$$

where w and z are uncorrelated Gaussian white noise with unit variance, and q is a Poisson counter that equals one with probability λ/δ. Here $1/\delta$ is the approximation to dt. The discrete process, equation 3.3.5, tends in distribution to the continuous process, equation 3.3.1, when $1/\delta$ goes to zero at a sufficient rate, see Guard (1988).

Daily returns

As before, define,

$$y_{t+1} \equiv \ln\left(\frac{S_{t+1}}{S_t}\right); \quad t = 1,2,...T$$

as the observed daily returns. In the simulated sample the daily return is,

$$y^s_{t+1} \equiv \sum_{j=t+1/\delta}^{t+\delta/\delta} y^s_j = (\ln S^s_{t+1/\delta} - \ln S^s_t) + (\ln S^s_{t+2/\delta} - \ln S^s_{t+1/\delta}) + ... + (\ln S^s_{t+1} - \ln S^s_{t+(\delta-1)/\delta})$$

the cumulative sum of the realizations in the interval [t+$1/\delta$, t+ δ/δ] We used $\delta = 5$, so the "daily" return equals the sum of 5 draws from the finer time process.

Algorithm

Auxiliary Model

1. Obtain the vector of estimates $\hat{\beta}_{1,T}$ from the maximization of the log likelihood function of the mixture of normals auxiliary model using the observed data.
2. Calculate and square the residuals.
3. Estimate the vector of parameters of the AR(10) auxiliary model $\hat{\beta}_{2,T}$, and code the scores.

Simulations

1. Simulate N x T x δ random variables to be used in the subsequent calculations, where δ is the discretization factor.
2. Choose: (i) initial values for the model's vector of parameters θ, and (ii) an arbitrary weighting matrix, $\tilde{I}_0$.

First Pass

1. Use random variables and θ to evaluate the scores.
2. Find a new θ_i that makes the value of the objective function smaller.
3. Iterate on 1-2 until the value of the score is as close to zero as possible, that is until convergence criterion is reached. This gives consistent parameter estimates, $\tilde{\theta}$.

Estimate the Optimal Weighting Matrix

Use the consistent parameter estimates, $\tilde{\theta}$, to estimate the optimal weighting matrix, $\hat{I}_0$.

Second Pass

1. Use random variables and the consistent estimates, $\tilde{\theta}$, to evaluate the scores.
2. Find a new $\tilde{\theta}_i$ that makes the value of the objective function smaller with the optimal weighting matrix smaller.
3. Iterate on 1-2 until the value of the score is as close to zero as possible, that is until convergence criterion is reached. This gives optimal parameter estimates, $\hat{\theta}$.

Results

We estimated the stochastic volatility jump diffusion process for the NOK/GBP returns, using 20 simulated series of length 2180.[8] We used a discretization factor of five, i.e., $\delta = 5$. The estimation procedure is very computer intensive. Each run took over an hour. The estimation of the stochastic volatility part seemed to be fairly robust since the estimation procedure converged to the same estimates for quite different starting values. The estimate of the diffusion mean was also stable for varying initial values. The jump-parameters μ_J, σ_J, and λ, however, frequently converged to different values for different starting values. Jumps, by definition, occur infrequently and are hard to identify. Our choice of starting values for the jump-parameters were based on qualitative reasoning on the number and size of jumps, and also on the size of the loss function. Estimation of this process needs good starting values to ensure convergence to a global minimum.

We performed a number of estimations with different starting values to search for a global minimum. The results of the estimation are summarized in Table 3.3.

TABLE 3.3

ESTIMATION OF A STOCHASTIC VOLATILITY JUMP DIFFUSION PROCESS

T = 2180, N = 20	a	b	c	μ	μ_J	σ_J	λ
NOK/GBP							
Parameter estimates	0.0333	0.0377	0.2635	0.0680	–0.5805	8.3802	0.0098
Standard error	0.0015	0.0012	0.0075	0.0050	0.5418	0.6506	0.0020

All of the coefficients are significant at the 5% level except for the mean of the jump process.

Matching the moments

The first four unconditional moments of the SVJD process can be calculated from the estimated model parameters and compared to the sample moments.

The mean of the SVJD process

$$\mu = \mu + \lambda\mu_J = 0.062$$

is larger than sample average of 0.04, but well within the two standard deviation confidence interval.

The unconditional variance equals the expected stochastic variance plus the contribution of the jump,

$$\sigma^2 = Eh^2 + \lambda(\mu_J{}^2 + \sigma_J{}^2)$$

The mean reverting specification for stochastic volatility in equation 3.3.1 implies that the log of h^2 is normally distributed,

$$\ln h^2 \sim N\left(\mu_{\ln h^2}, \frac{c^2}{2b}\right)$$

So the expected value of the stochastic variance is,

$$Eh^2 = \mu_{\ln h^2} + \frac{c^2}{4b} = 3.98$$

And the variance of the SVJD process is,

$$\sigma^2 = 4.66,$$

within the two-standard deviation confidence interval for the sample unconditional variance, 4.86, in Table 2.1.

The skewness calculated from the SVJD model,

$$\text{Skewness} = -0.12,$$

is not very close to the sample skewness of 0.82.

The SVJD generates 90% of the sample excess kurtosis,

$$\text{Excess Kurtosis} = 9.72,$$

but, it is still below the two-standard deviation confidence interval for the unconditional estimate in Table 2.1.

Specification test

Gallant and Tauchen (1996) and Gourieroux, Monfort, and Renault (1993) show that under the null hypothesis that the underlying model is correctly specified the scaled value of the objective function,

$$\xi = T\min_{\theta}\left[\sum_{s=1}^{N}\sum_{t=1}^{T} l_{\beta}(y^s{}_t(\theta);\hat{\beta}_T)\right]' I_0^{-1}\left[\sum_{s=1}^{N}\sum_{t=1}^{T} l_{\beta}(y^s{}_t(\theta);\hat{\beta}_T)\right] \quad 3.3.6$$

is distributed asymptotically as $\chi^2(q-p)$. Here q = dim(β) –is the number parameters in the auxiliary model, and p = dim(θ) – is the number of parameters in the underlying model.

The SVJD model is not rejected at standard confidence levels using the χ^2 test. The scaled minimized value of the loss is $\xi = 13.71$. Our auxiliary model has 17 parameters and the underlying SVJD model has 7 parameters, which leaves 10 degrees of freedom. The P value for the cdf, $\chi^2(13.7, 10) = 0.81$, shows that almost twenty percent of the time higher realizations occur by chance.

IV. Monte Carlo

As an informal check on the estimates in Section III we did Monte Carlo experiments. We investigated two questions: (1) (a) how does the distribution of the estimator with a sample of 2000 compare to its asymptotic distribution, and (b) is the size of the specification test from the asymptotic distribution approximately correct for the small sample? And (2) if the data generation process is a SVJD process do estimates of misspecified jump-diffusion model ignore the jumps and try to represent the volatility clustering with two regimes? It turns out the answers to these questions are (1) (a) the sample size seems to be adequate for estimates of stochastic volatility, but not for the estimates of the parameters of the jump distribution, and (b) the misspecification test rejects too frequently in the small sample. And (2) the estimates from a misspecified jump-diffusion model ignore the jump process and estimate instead high and low volatility regimes.

4.1 Experiment

The Monte Carlo allow us to control the environment. We parameterized the SVJD model as follows,

a	b	c	μ	μ_J	σ_J	α
0	0.06	0.28	0	2	7.5	0.01

$$\begin{aligned}\ln\left(\frac{S_{j+1}}{S_j}\right) &= \frac{\mu_d}{\delta} + h_{j+1}w_{j+1}\sqrt{1/\delta} + kq_{j+1}\\ \ln\left(\frac{h_{j+1}^2}{h_j^2}\right) &= \frac{a}{\delta} - \frac{b}{\delta}\ln h_j^2 + cz_{j+1}\sqrt{1/\delta}\end{aligned} \quad 4.1$$

We choose a positively skewed jump process with infrequent (1% on average) large jumps. The stochastic volatility process is noisy and reverts slowly to the mean. We set the drift parameters (μ, μ_J) to zero. These are basically nuisance scale parameters.

We generated 30 samples[9] of size 10,000 which yields 2000[10] "daily" observations using a δ of 5. For each sample we estimated the correctly specified SVJD, did a misspecification test for the model, and estimated the misspecified JD process.

4.2 Model parameter estimates

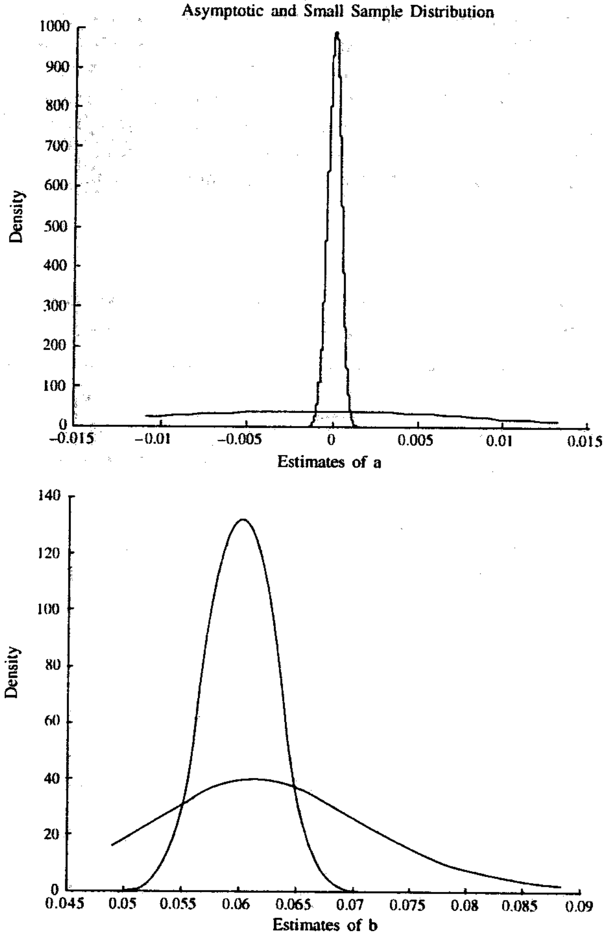
Table A1 in the Appendix records the results of the EMM estimates of the parameters of the SVJD model. Table A2 in the Appendix records the results of the maximum likelihood estimates of the parameters of the JD model.

Figures 4.1 and 4.2 present the data graphically. Each graph shows the density of the parameter estimator calculated from the 30 Monte Carlo draws –the "small sample" density[11]– and the asymptotic density.

FIGURE 4.1

DENSITIES FOR STOCHASTIC VOLATILITY ESTIMATORS

Asymptotic and Small Sample Distribution



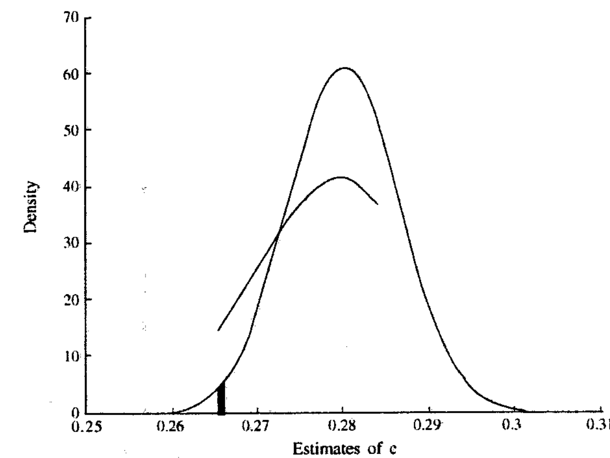


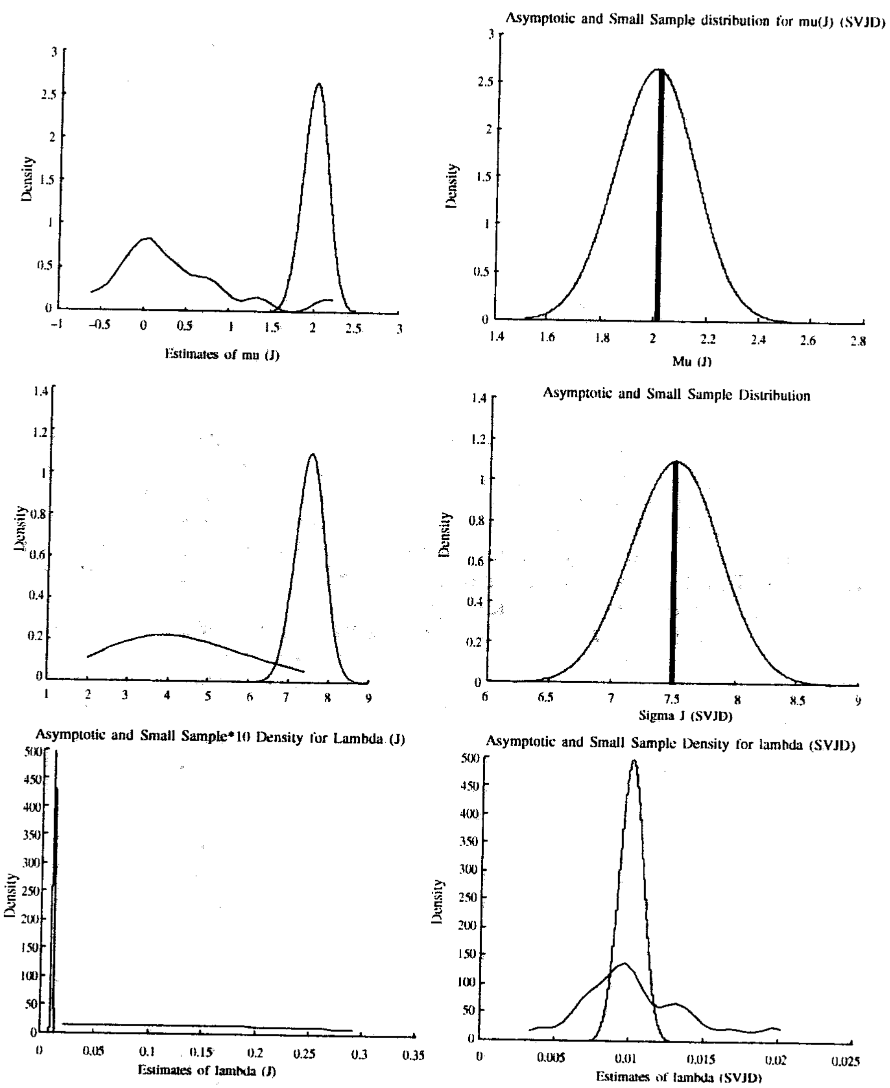
Figure 4.1 shows the densities of the estimators for the stochastic volatility equation. The graphs with the high peaks centered at the true values are the asymptotic densities of the parameter estimates. The small sample densities are centered at the true values and are more diffuse, but the errors are not large. When the model is correctly specified 2000 observations seem sufficient to get reasonable EMM estimates of the stochastic volatility process although the asymptotic confidence intervals are too small.

Figure 4.2 is much more dramatic. It shows the densities of the estimators of the jump-diffusion equation process for the correctly specified – stochastic volatility-jump diffusion model –and the misspecified– jump diffusion model. The right panels show the densities of the EMM parameter estimator for the correctly specified process. The left panels show the densities of the maximum likelihood estimator for the misspecified model. The graphs in the left panel vividly confirm our conjecture, the misspecified jump diffusion model essentially splits the data into two regimes: it labels the high volatility regime the jump and the low volatility the GBM process. The graphs in the right panel indicate that 2000 observations are not enough to pin down the parameters of the jump distribution in the correctly specified SVJD model. Jumps are unusual events by definition. On average there are only 20 draws from the jump distribution.

The top panels show the densities for the estimates of the mean of the jump process. The asymptotic density is centered at the true mean of two. In the left panel the mode of the density for the misspecified jump diffusion model is zero. Almost the entire density lies to the left of the true value of two. The JD estimator maximizes the likelihood by splitting the sample into high and low volatility regimes. The small sample density for the EMM estimator in the right panel is too good to be believed. All the estimates fall in the dark line centered at two.

FIGURE 4.2

DENSITIES FOR JUMP-DIFFUSIONS



The small sample density is much less diffuse than the asymptotic density. Something is wrong with the Monte Carlo design. Most likely it is the starting values. We used the true values as initial values in both the nonlinear estimation procedures. The EMM algorithm never moved from the starting values. In future work we will randomize the initial values which will probably give a very diffuse density.

The middle panel shows the estimator of the volatility of the jump process. The true volatility is 7.5. The misspecified jump diffusion model badly underestimates the true volatility. The mode of the density is centered at four and most of the density lies to the left of the true value. The right panel shows to EMM estimates too tightly grouped around the true value. Again the EMM algorithm remained at the initial conditions.

The bottom panel shows the estimates of jump probability. The misspecified JD model shown in the left panel badly overestimates the jump probability. The true probability is 1%. All of the estimates are greater than 1% and the average is 7.5%. The SVJD model is shown in the right-hand panel. The mode of the density is slightly less than the correct value of 1%. The density is more diffuse than the asymptotic density, but most of the estimates are between 1/2% to 1 1/2%. It seems that 2000 observations are enough to get reasonable estimates of the probability of a jump (unusual event), but not to pin down the parameters of the jump distribution.

4.3 Misspecification test

The scaled minimized value of the loss function,

$$\xi = T \min_\theta \left[\sum_{s=1}^{N} \sum_{t=1}^{T} l_\beta(y^s{}_t(\theta); \hat{\beta}_T) \right]' I_0^{-1} \left[\sum_{s=1}^{N} \sum_{t=1}^{T} l_\beta(y^s{}_t(\theta); \hat{\beta}_T) \right]$$

is asymptotically distributed $\chi^2(q-p)$. Here q = 17 –the number of parameters in the auxiliary model, and p = 7 –the number of parameters in the underlying model.

The critical values are,

$$\chi^2_{(1-\alpha,10)} = \chi^2_{(0.90,10)} = 15.99$$

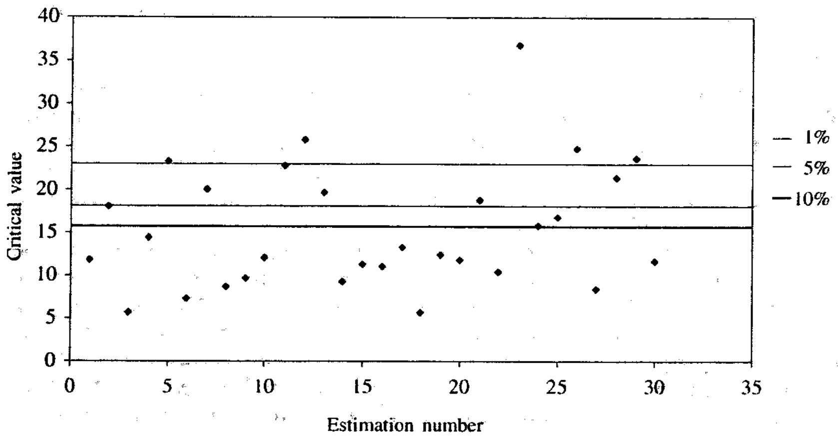
$$\chi^2_{(0.95,10)} = 18.31$$

$$\chi^2_{(0.99,10)} = 23.21$$

Figure 4.3 shows the realizations in 30 trials and the asymptotic critical values.

FIGURE 4.3

MISSPECIFICATION TEST OF THE SVJD-PROCESS



Number of parameters in underlying model: 7
Number of parameters in auxiliary model: 17
Degrees of freedom: 17 – 7 = 10
Number of rejections at the 10% level: 12 => 40.0%.
Number of rejections at the 5% level: 10 => 33.3%.
Number of rejections at the 1% level: 4 => 16.7%.

The specification test rejects to frequently.[12] The test rejected 40% of the time at the 10% significance level, 33.3% of the time at the 5% level, and 16.7% of the time at the 1% level.

V. Conclusions

High frequency financial returns data display potential jumps, volatility clustering, skewness, and excess kurtosis. The goal is to find a parsimoniously parameterized model that captures the essential features in the data. The main results of the paper are: (1) Capturing these features requires a specification that allows both jumps and stochastic volatility. (2) A specification that only allows for jumps badly misrepresents the data. And (3) reasonably accurate estimates of the parameters of the jump distribution require a very large sample.

This paper uses a simulation-based technique to estimate a stochastic-volatility jump-diffusion (SVJD) model of the Norwegian-British exchange rate. Simulation-based estimation is a relatively new, flexible, and general technique, but it is computationally intensive. We also estimated the traditional geometric Brownian motion (GBM) and GBM plus a Poisson jump (JD) model by maximum likelihood. The SVJD model nests the GBM and JD specifications.

We analyze the results by examining how well the models match the first four unconditional moments of the data and analyze the technique with Monte Carlo experiments. The GBM specification is distributed normally iid. It has no skewness or excess kurtosis. The JD specification is also distributed iid, but it can be skewed and it has excess kurtosis. The conditional and unconditional moments of iid distributions are the same, so they cannot match the conditional moments (volatility clustering.) The SVJD specification could match the conditional and unconditional moments.

Our estimation results, Section III, show that the JD and SVJD models match three of the four unconditional moments pretty well. Neither model matches the positive skewness in the data. The JD model generates 62% of the excess kurtosis and the SVJD model generates 90% of the excess kurtosis. The parameter estimates, however, seem to indicate a serious flaw in the JD model that is confirmed by the Monte Carlo experiments. Jumps in financial markets, by definition, are infrequent events. Volatility clustering is normal. The JD model fits volatility clustering by segregating the data into a high volatility regime –the jump regime– and a low volatility regime. It ignores the infrequent jumps. The estimated probability of a jump for any day is 1/3 in the JD model. The parameter estimates for the SVJD model are more reasonable –the probability of a jump is about 1%. A formal test does not reject the SVJD model.

The Monte Carlo experiments in Section IV confirm our conjecture that if the data generation process has both stochastic volatility and jumps, then a jump diffusion model is a very bad approximation. The Monte Carlo experiments also show that 2000 observations, which is a fairly large sample for most economic applications, is not enough to pin down the parameters of the distribution of unusual events.

APPENDIX

TABLE A1

MONTE CARLO PARAMETER ESTIMATES FOR SVJD MODEL

Simulation #		a	B=1−b	c	μ	μ_J	σ_J	λ
	True values	0	0.94	0.28	0	2	7.5	0.01
1		−0.0025	0.9422	0.2812	−0.0004	1.9894	7.5102	0.0132
2		0.0094	0.9322	0.2769	−0.0002	2.0003	7.5001	0.0099
3		−0.0077	0.9381	0.2793	0.0004	2.0001	7.5001	0.0073
4		−0.0116	0.9397	0.2802	0.0003	2.0001	7.5000	0.0073
5		−0.0003	0.9400	0.2799	0.0000	2.0001	7.5000	0.0101
6		0.0024	0.9403	0.2801	0.0000	2.0001	7.5001	0.0093
7		−0.0050	0.9350	0.2746	−0.0001	1.9996	7.4971	0.0106
8		0.0061	0.9396	0.2800	0.0002	1.9998	7.5003	0.0033
9		0.0071	0.9187	0.2749	−0.0024	2.0008	7.5002	0.0088
10		−0.0065	0.9404	0.2800	0.0002	2.0000	7.5001	0.0087
11		0.0067	0.9295	0.2760	−0.0013	1.9997	7.5001	0.0067
12		−0.0005	0.9388	0.2793	0.0000	1.9999	7.5001	0.0077
13		0.0002	0.9284	0.2743	−0.0032	1.9992	7.5000	0.0126
14		0.0023	0.9407	0.2802	−0.0002	2.0000	7.5000	0.0096
15		−0.0074	0.9296	0.2768	0.0006	1.9969	7.5023	0.0146
16		−0.0122	0.9390	0.2795	0.0017	2.0001	7.4998	0.0111
17		−0.0090	0.9289	0.2749	−0.0028	2.0002	7.5000	0.0137
18		−0.0045	0.9398	0.2800	−0.0002	1.9999	7.5000	0.0100
19		0.0016	0.9436	0.2816	−0.0017	1.9996	7.5000	0.0098
20		−0.0006	0.9387	0.2795	−0.0001	2.0000	7.5001	0.0047
21		−0.0070	0.9268	0.2759	−0.0022	1.9999	7.5001	0.0132
22		−0.0059	0.9374	0.2789	0.0001	1.9998	7.5000	0.0078
23		0.0014	0.9408	0.2801	−0.0003	2.0001	7.5000	0.0163
24		−0.0029	0.9363	0.2757	−0.0009	1.9998	7.5000	0.0099
25		−0.0160	0.9386	0.2799	0.0001	2.0000	7.4999	0.0196
26		0.0153	0.9391	0.2797	0.0002	2.0000	7.4999	0.0201
27		0.0047	0.9419	0.2809	0.0004	1.9997	7.5000	0.0093
28		−0.0011	0.9509	0.2834	−0.0015	1.9996	7.5000	0.0175
29		−0.0069	0.9402	0.2801	−0.0001	1.9999	7.4999	0.0126
30		0.0132	0.9432	0.2808	−0.0013	2.0002	7.5000	0.0065
	Average	−0.0012	0.9373	0.2788	−0.0005	1.9995	7.5003	0.0107
	Standard dev	0.0074	0.0063	0.0024	0.0011	0.0020	0.0020	0.0040

Asymptotic Distribution	θ_{0i}	$W(N,I_0)_{ii}^{1/2}$
a	0.00	0.0004
B = 1− b	0.94	0.0030
c	0.28	0.0065
μ	0.00	0.0028
μ_J	2.00	0.1507
σ_J	7.50	0.3631
λ	0.01	0.0008

TABLE A2

JUMP DIFFUSION ESTIMATES

Simulation		μ	σ	μ_J	σ_J	λ
	True Values	0		2.0	7.5	0.01
1		0.026	0.894	0.141	2.606	0.118
2		−0.005	0.883	0.029	1.992	0.291
3		0.016	0.918	0.034	3.216	0.070
4		0.007	0.939	0.256	2.895	0.101
5		0.001	1.067	0.019	5.353	0.049
6		−0.008	0.966	1.306	4.396	0.050
7		−0.025	1.064	−0.048	4.705	0.034
8		−0.004	0.960	−0.595	4.316	0.053
9		0.017	1.030	−0.222	3.330	0.096
10		−0.019	1.063	−0.631	6.041	0.036
11		0.012	0.967	−0.264	2.511	0.139
12		0.015	1.075	2.101	6.449	0.028
13		0.000	0.927	−0.347	3.145	0.124
14		0.032	1.056	0.707	4.226	0.049
15		−0.030	1.051	0.262	6.045	0.041
16		−0.001	0.966	0.025	3.377	0.090
17		−0.025	1.041	−0.158	4.781	0.050
18		−0.015	0.982	0.757	4.603	0.046
19		0.031	0.904	0.443	3.643	0.062
20		−0.022	0.995	1.367	6.778	0.034
21		−0.011	0.991	0.833	4.736	0.050
22		−0.027	0.979	0.452	4.015	0.082
23		−0.022	1.133	2.222	7.423	0.020
24		−0.031	0.922	0.673	2.854	0.081
25		0.015	1.036	−0.249	4.060	0.050
26		0.004	1.022	0.412	4.634	0.049
27		−0.013	0.962	−0.010	3.412	0.122
28		0.029	1.082	0.942	5.952	0.035
29		−0.022	0.930	0.064	2.991	0.090
30		−0.081	1.123	0.310	3.904	0.074
Average		−0.005	0.998	0.361	4.280	0.074
Standard Deviation		0.025	0.024	0.480	0.290	0.013

Notes

1 The exchange rate is the price of a British Pound in Norwegian Kroner.
2 The data are multiplied by 10^3.
3 The description of the geometric Brownian motion and jump-diffusion model closely follows Das and Sundaram's (1999) extremely clear exposition.
4 In principle an infinite number of jumps could occur during the day. We set the maximum number of jumps per day, Q, at ten which seems reasonable. Bates and Craine (1999), and Jorion (1988) use a maximum of ten for daily data. Jorian presents some evidence that ten is sufficient.
5 This section is based on Gourieroux and Monfort, Chapter 4.
6 Precisely the matrix $D'I_0^{-1}D$ must have rank equal to the number of elements in the parameter vector θ.
7 Monfardini (1998) experimented with auxiliary models for mean-reverting stochastic volatility processes and found that an AR(10) worked well.
8 The 10 year sample has 2180 observations.
9 Thirty draws are few to draw any strong inferences from. Each run, however, takes several hours which is why we have only 30 draws.
10 The data have 2180 observations.
11 The smoothed small sample densities were calculated using a normal kernal estimator.
12 Monfardini's more comprehensive Monte Carlo study also finds the test rejects to frequently in small samples.

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